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ON THE LOGARITHMIC-SINGULARITY CORRECTION IN THE KERNEL FUNCTION METHOD OF SUBSONIC LIFTING-SURFACE THEORY

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ON THE LOGARITHMIC-SINGULARITY CORRECTION IN THE KERNEL FUNCTION METHOD OF SUBSONIC LIFTING-SURFACE THEORY

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SUMMARY

A new logarithmic-singularity correction factor is derived for use in kernel function methods associated with Multhopp's subsonic lifting-surface theory. Because of the form of the factor, a relation has been formulated between the numbers of chordwise and spanwise control points needed for good accuracy. This formulation is developed and discussed. Numerical results are given to show the improvement of the computation with the new correction factor.

INTRODUCTION

There are various kinds of singularities which arise in the kernel function methods of the lifting-surface theory. Since Multhopp published his method in 1950 (ref. 1), much effort has been expended in obtaining an accurate accounting of these singularities. The singularities are associated with: (1) the mathematical modeling of the lift potential of a wing surface (i.e., doublets) and the resulting downwash produced by them as calculated in the analytical or numerical spanwise integration over the lifting surface, (2) leading-edge pressures, and (3) changes in wing sweep at the apex or along the span (refs. 2 and 3).

Singularities associated with wing sweep are not given special treatment (i.e., special pressure modes), but instead Multhopp's procedure for rounding the leading and trailing edges in the vicinity of the affected regions is used (ref. 1). Leading-edge pressure singularities are expected because of the modes assumed, but they add to the first singularity problem listed and are treated in the context of this problem. The first singularity is the main subject considered herein.

The spanwise integration over the lifting surface generally has associated with it a logarithmic-type singularity term at some spanwise position which in a purely numerical

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treatment can lead to significant errors. To minimize these errors, the usual procedure is to separate the logarithmic term and evaluate it to provide a "correction" to the remainder of the integral. (Refs. 1 and 4 are examples.) The term which results is called the logarithmic-singularity correction (LSC). This approach was introduced by Multhopp in reference 1 (used later by others in refs. 2 to 9) along with a procedure for the LSC calculation. Improvements in the computational procedure have been formulated by Mangler and Spencer (ref. 9) and Zandbergen, Labrujere, and Wouters (ref. 2). However, the most commonly used correction has a serious deficiency which arises because the correction term consists only of the leading term of a correction series which diverges as the chordwise control points approach the leading and trailing edges. The exclusion of the high-order correction terms, logarithmic in nature, in this divergent series has been credited by Jordan (ref. 8) as the main cause of significant errors in some existing lifting-surface methods when the number of chordwise control points is increased. Jordan (ref. 8), Wagner (ref. 5), and Medan (ref. 3) have advocated the use of wing-edge control points for better accuracy in the solution. These wing-edge control points are known to be associated with algebraic, instead of logarithmic, singularity corrections. However, even if wing-edge control points are used, the logarithmic correction factors which are needed for interior control points are still inaccurate near the edges.

Another feature of existing kernel function methods is that the relation between numbers of chordwise and spanwise control points used to obtain a reliable solution has been empirical, derived by applying the theory to a limited number of planforms (ref. 7). However, experience indicates that this empirical relation is not applicable to arbitrary combinations of configurations and Mach number. It is in fact highly dependent on planform and Mach number.

In this report, a method of obtaining a convergent correction series is presented and the leading term of such a series is derived. Better convergence characteristics of a modified Multhopp's method are illustrated when the new correction factor is used. In addition, a general relation governing numbers of chordwise and spanwise control points is derived and demonstrated to provide accurate solutions for arbitrary planar planforms.

Some details of the development are given in appendixes A and B, and implementation of the analysis is given in appendix C.

SYMBOLS

A $= \sqrt{(1 - X)^2 + Y^2}$

B $= \sqrt{X^2 + Y^2}$

b	wing span, equal to 2 in this report
$b_{\nu\nu'}, b_{\mu n}$	Multhopp's quadrature weighting factors defined in equation (B21)
$\frac{C_{D,ii}}{C_{D,i}}$	ratio of near-field to far-field vortex drag
C_{L_α}	overall wing lift-curve slope
ΔC_p	lifting pressure coefficient
C_1	constant to be determined
$c(\eta), c(y)$	streamwise half-chord at η and y, respectively, nondimensionalized by $b/2$
c_{av}	average chord
$E(k)$	complete elliptic integral of second kind
$E(\varphi,k)$	incomplete elliptic integral of second kind
$F(\varphi,k)$	incomplete elliptic integral of first kind
$G_j(\phi)$	function of ϕ defined by equation (21)
h_j	jth chordwise loading function
I_j	jth chordwise integral defined by equation (6)
$I_{j\nu\nu'} I_{j\mu n}$	see equation (B22)
J_1, J_2	integrals defined by equations (15) and (16)
$K(k)$	complete elliptic integral of first kind
k^2	modulus of elliptic integrals squared, $\frac{1 - (A - B)^2}{4AB}$

k'^2 complementary modulus of elliptic integrals squared, $1 - k^2 = \frac{(A + B)^2 - 1}{4AB}$

$k'_{\nu\nu}, k'_{\nu n}$ see equation (B20)

k'_1, k'_2 k' at left and right wing tip, respectively

L_j logarithmic component of I_j

M free-stream Mach number

m number of span stations where pressure modes are defined

N number of chordwise control points at each of m span stations

q_j jth spanwise loading function

w perturbation velocity in vertical direction nondimensionalized with respect to free-stream velocity

X $= \frac{x - \xi_{le}(\eta)}{2\, c(\eta)}$

X_0 $= \frac{x - \xi_{le}(y)}{2\, c(y)}$

X_1 $= \frac{\xi - \xi_{le}(\eta)}{2\, c(\eta)}$

Y $= \beta \frac{y - \eta}{2\, c(\eta)}$

Y_0 $= \beta \frac{y - \eta}{2\, c(y)}$

x,y streamwise and spanwise rectangular Cartesian coordinates nondimensionalized with respect to b/2 and associated with control-point locations

x_{ac} aerodynamic center in fractions of reference chord and referenced to leading edge of reference chord, positive aft

β $= \sqrt{1 - M^2}$

ϵ small positive number

θ angle for locating control points along span, $\cos^{-1}(\eta)$

Λ sweep angle, deg

$\Lambda_0(\psi,k)$ Heuman's lambda function, equation (B3)

λ taper ratio, $\frac{\text{Tip chord}}{\text{Root chord}}$; also index in equation (23)

ξ,η rectangular Cartesian coordinates, nondimensionalized with respect to b/2, in x- and y-directions, respectively, and associated with pressure doublet locations

$\Pi(\alpha^2,k)$ complete elliptic integral of third kind

$\Pi(\varphi,\alpha^2,k)$ incomplete elliptic integral of third kind

ϕ angular coordinate defined by $X = \frac{1}{2}(1 - \cos\phi)$

ϕ_1 angular coordinate defined by $X_1 = \frac{1}{2}(1 - \cos\phi_1)$

ψ $= \sin^{-1}(A - B)$

Subscripts:

le leading edge

n spanwise influencing station associated with particular η values

s chordwise control point associated with particular ϕ values

te trailing edge

ν spanwise control point associated with particular y values

Abbreviation:

LSC logarithmic-singularity correction term

The symbol $\oint$ denotes finite-part integration.

MATHEMATICAL DEVELOPMENT

In linearized steady subsonic flow theory for a thin wing, the lifting pressure ΔC_p is related to the downwash distribution on the wing surface through the following singular integral equation:

$$w(x,y) = \frac{1}{8\pi} \oint_{-1}^{1} \int_{\xi_{le}(\eta)}^{\xi_{te}(\eta)} \frac{\Delta C_p(\xi,\eta)}{(y-\eta)^2} \left[1 + \frac{x-\xi}{\sqrt{(x-\xi)^2 + \beta^2(y-\eta)^2}} \right] d\xi \, d\eta \tag{1}$$

where the spanwise integration is defined by the finite-part concept. By introducing the new variables,

$$X = \frac{x - \xi_{le}(\eta)}{2\,c(\eta)} \qquad X_1 = \frac{\xi - \xi_{le}(\eta)}{2\,c(\eta)} \tag{2}$$

equation (1) becomes

$$w(X,y) = \frac{1}{4\pi} \oint_{-1}^{1} \frac{1}{(y-\eta)^2} \int_{0}^{1} \Delta C_p(X_1,\eta) \times \left(1 + \frac{X - X_1}{\sqrt{(X-X_1)^2 + \beta^2 \left\{ (y-\eta)/\left[2\,c(\eta)\right] \right\}^2}} \right) c(\eta)\, dX_1\, d\eta \tag{3}$$

In equation (3), the lifting pressure coefficient is usually approximated by

$$\Delta C_p(X_1,\eta) = \frac{2}{\pi} \sum_{j=0}^{N-1} \frac{q_j(\eta)}{c(\eta)} h_j(\phi_1) \tag{4a}$$

where

$$h_j(\phi_1) = \frac{\cos j\phi_1 + \cos (j+1)\phi_1}{\sin \phi_1} \tag{4b}$$

and

$$X_1 = \frac{1}{2}\left(1 - \cos \phi_1\right) \tag{4c}$$

It follows that equation (3) can be written as

$$w(X,y) = \frac{1}{4\pi} \int_{-1}^{1} \frac{1}{(y-\eta)^2} \left[\sum_{j=0}^{N-1} q_j(\eta)\, I_j(X,Y,\eta) \right] d\eta \tag{5}$$

where

$$I_j(X,Y,\eta) = \frac{2}{\pi} \int_0^1 \frac{\cos j\phi_1 + \cos (j+1)\phi_1}{\sin \phi_1} \left[1 + \frac{X - X_1}{\sqrt{(X-X_1)^2 + Y^2}} \right] dX_1 \tag{6}$$

and

$$Y = \beta \frac{y - \eta}{2\, c(\eta)}$$

Now, it has been shown (ref. 1) that I_j behaves like $(y-\eta)^2 \ln|y-\eta|$ near $\eta = y$. The factor $(y-\eta)^2$ is canceled by the same factor in the denominator of equation (5), so that a logarithmic singularity at $\eta = y$ appears. Accurate spanwise integration can be done only if this logarithmic singularity is carefully accounted for.

Mangler and Spencer (ref. 9) removed the logarithmic component of I_j, called L_j, before performing the spanwise integration. This approach, also used here, leads to (eq. (B14))

$$I_j(X,Y,\eta) = I_j^*(X,Y,\eta) + L_j \tag{7}$$

so that the term in the summation in equation (5) can be replaced with (eq. (B15))

$$q_j(\eta)\, I_j(X,Y,\eta) = q_j(y)\left(I_j - I_j^*\right) + \left[q_j(\eta) - q_j(y)\right] I_j + q_j(y)\, I_j^* \tag{8}$$

Obviously, equation (8) involves L_j and therefore requires attention. As $\eta \to y$, $I_j^*(X,Y,\eta)$ varies in X like $I_j(X,Y,y)$ and thus

$$L_j = \begin{matrix}\text{Logarithmic} \\ \text{component} \\ \text{of}\end{matrix} \lim_{\eta \to y} \left\{I_j(X,Y,\eta) - I_j(X,Y,y) = \Delta I_j\right\} \tag{9}$$

So

$$\Delta I_j = \frac{2}{\pi}\left\{\int_0^X \frac{\cos j\phi_1 + \cos (j+1)\phi_1}{\sin \phi_1}\left[-1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1\right.$$

$$\left. + \int_X^1 \frac{\cos j\phi_1 + \cos (j+1)\phi_1}{\sin \phi_1}\left[1 - \frac{X_1 - X}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1\right\}$$

It is observed that for small Y, the factors inside the brackets in this equation are small when X_1 is not close to X and have significant magnitude only near $X_1 = X$. For this reason, Multhopp (ref. 1) argued that for deriving the logarithmic correction term, it is reasonable to develop the function h_j (eq. (4b)) into a Taylor series about the point $X_1 = X$, or equivalently about $\phi_1 = \phi$. The result can be integrated exactly and then developed into a series for small Y to obtain the logarithmic components. Now, it is obvious that h_j does not have a Taylor series expansion about the integration end points, $X_1 = 0$ and $X_1 = 1$. For example, by using equation (4c), the Taylor series expansion for h_0 is obtained as

$$h_0(X_1) = h_0(X) + \frac{dh_0(X)}{dX_1}(X_1 - X) + \frac{1}{2}\frac{d^2h_0(X)}{dX_1^2}(X_1 - X)^2 + \ldots$$

$$= \frac{(1 - X)^{1/2}}{X^{1/2}} - \frac{X_1 - X}{2X^{3/2}(1 - X)^{1/2}} + \frac{(3 - 4X)(X_1 - X)^2}{8X^{5/2}(1 - X)^{3/2}} + \ldots \tag{10}$$

This series obviously does not converge near $X = 0$ and $X = 1$. This nonconvergence of the series expansion is due to the presence of $\sin \phi_1 = 2\sqrt{X_1(1 - X_1)}$ in the

denominator of h_j. Therefore, a natural way to improve the series expansion is to expand only $\cos j\phi_1 + \cos (j+1)\phi_1$ as follows:

$$\cos j\phi_1 + \cos (j+1)\phi_1 = \cos j\phi + \cos (j+1)\phi + \left\{\frac{d}{d\phi_1}\left[\cos j\phi_1 + \cos (j+1)\phi_1\right]\frac{d\phi_1}{dX_1}\right\}\Bigg|_{\phi_1=\phi} (X_1 - X) + O(X_1 - X)^2$$

$$= g_1(\phi) + g_2(\phi)(X_1 - X) + O(X_1 - X)^2 \tag{11}$$

where

$$g_1(\phi) = \cos j\phi + \cos (j+1)\phi \tag{12}$$

and

$$g_2(\phi) = -2\,\frac{j \sin j\phi + (j+1)\sin (j+1)\phi}{\sin \phi} \tag{13}$$

Hence, for small Y, equation (9) can be written as

$$\Delta I_j = \frac{1}{\pi}\int_0^1 \frac{g_1(\phi) + g_2(\phi)(X_1 - X)}{\sqrt{X_1(1 - X_1)}}\left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1 - I_j'(X,Y,y)$$

$$= \frac{1}{\pi}\left[g_1(\phi) + (1 - X)g_2(\phi)\right]\int_0^1 \frac{1}{\sqrt{X_1(1 - X_1)}}\left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1$$

$$- \frac{g_2(\phi)}{\pi}\int_0^1 \sqrt{\frac{1 - X_1}{X_1}}\left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1 - I_j'(X,Y,y) \qquad \left(|Y| \ll 1\right) \tag{14}$$

where equation (6) has been used and $I_j^*(X,Y,y)$ is the integral $I_j(X,Y,y)$ with the loading function expanded in accordance with equation (11). Obviously, the logarithmic correction term comes only from the integrals expressed in equation (14). Note that if higher order correction terms are desired, more terms should be retained in equation (11). The two integrals of equation (14) can be integrated exactly to give

$$J_1 = \int_0^1 \sqrt{\frac{1 - X_1}{X_1}} \left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}} \right] dX_1$$

$$= \frac{\pi}{2} \left\{ 1 - \Lambda_0(\psi,k) + \frac{4}{\pi} \sqrt{AB} \left[E(k) - k'^2 K(k) \right] \right\} \tag{15}$$

$$J_2 = \int_0^1 \frac{1}{\sqrt{X_1(1 - X_1)}} \left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}} \right] dX_1$$

$$= \pi \left[1 - \Lambda_0(\psi,k) \right] \tag{16}$$

Details of the integration are given in appendix A.

In appendix B, the LSC is derived; a summary of the derivation is given here. The relation between Y and k' is shown to be (eq. (B2))

$$Y^2 = 4k'^2 X^2 (1 - X)^2 \left\{ 1 + k'^2 \left[3 - 8X(1 - X) \right] \right\} + O\left(k'^6\right) \tag{17}$$

Equation (17) shows that small k' implies always small Y. However, small Y does not always imply small k', depending on the value of X. It follows that it is more natural to develop J_1 and J_2 for small k', rather than for small Y, to find the logarithmic terms. This choice represents a deviation from Multhopp's original method. In developing equations (15) and (16) for small k', it is found that the logarithmic components are (eqs. (B11) and (B10))

$$\text{Logarithmic component of } J_i = -\text{sign}(y-\eta)\,\frac{\beta}{2\,c(y)}\,\frac{1+\cos\phi}{\sin\phi}\,(y-\eta)^2\,\frac{dk'}{d\eta}\,\ln k' + O\left(k'^3 \ln k'\right) \tag{18}$$

$$\text{Logarithmic component of } J_2 = -\text{sign}(y-\eta)\,\frac{\beta}{c(y)}\,\frac{\cos\phi}{\sin\phi}\,(y-\eta)^2\,\frac{dk'}{d\eta}\,\ln k' + O\left(k'^3 \ln k'\right) \tag{19}$$

It follows from equations (14) to (16) and (18) to (19) that for small Y the dominating logarithmic component of I_j can be written as (eq. (B12))

$$L_j = -\text{sign}(y-\eta)\,\frac{\beta}{\pi}\,\frac{(y-\eta)^2\,\frac{dk'}{d\eta}\,\ln k'}{c(y)}\,G_j(\phi) \tag{20}$$

where (eq. (B13))

$$G_j(\phi) = \frac{\cos j\phi + \cos(j+1)\phi}{\sin\phi}\cos\phi + j\sin j\phi + (j+1)\sin(j+1)\phi \tag{21}$$

It is shown in appendix B that equation (20) can be reduced to the conventional form by expanding $dk'/d\eta$ and k' for small Y.

From the preceding development the first term of equation (8) involves L_j and contributes the following result to equation (5) (eq. (B16)):

$$-2\pi\, w\left(\phi_s, y_\nu\right)_{\text{part 1}} = -\sum_{j=0}^{N-1} \frac{\beta\, q_j\left(y_\nu\right)}{2\pi\, c\left(y_\nu\right)}\, G_j\left(\phi_s\right)\left(k'_1 \ln k'_1 - k'_1 + k'_2 \ln k'_2 - k'_2\right) \tag{22}$$

where k'_1 and k'_2 are the values of k' at the left and the right wing tip, respectively. The notation has been made compatible with Multhopp's notation (ref. 1). Reference 7 gives a detailed expianation of the subscripts which have been introduced. For the second and third terms of equation (8), Multhopp's method of integration can be applied by using the trigonometric interpolation formula of the type

$$q_j(\eta)\, I_j(\eta) = \frac{2}{m+1} \sum_{n=-\frac{m-1}{2}}^{\frac{m-1}{2}} \left(q_j I_j\right)_n \sum_{\lambda=1}^{m} \sin\lambda\theta_n \sin\lambda\theta \tag{23}$$

where $\theta_n = \frac{\pi}{2} - \frac{n\pi}{m+1}$ and m is the number of spanwise integration stations over the whole wing. The integration of the second and third terms of equation (8) in equation (5) results in (eq. (B19))

$$-2\pi\, w(\phi_s, y_\nu)_{\text{part 2}} = \sum_{j=0}^{N-1} \left\{ b_{\nu\nu}\, q_j(y_\nu)\, \pi I_{j\nu\nu} - \sum_n{}' \, b_{\nu n} \left[q_j(\eta_n)\, \pi I_{j\nu n} + \operatorname{sign}(y_\nu - \eta_n)\, \beta\, q_j(y_\nu)\, \frac{G_j(\phi_s)}{c(y_\nu)} (y_\nu - \eta_n)^2 \frac{dk'_{\nu n}}{d\eta} \ln k'_{\nu n} \right] \right\} \tag{24}$$

where the prime on the summation sign means that the term with $n = \nu$ is omitted from the summation. Combining equations (22) and (24) reduces equation (5) to the following equivalent algebraic equation:

$$-2\pi\, w(\phi_s, y_\nu) = \sum_{j=0}^{N-1} \left\{ q_j(y_\nu) \left[b_{\nu\nu}\pi I_{j\nu\nu} - \beta \frac{G_j(\phi_s)}{2\pi\, c(y_\nu)} \left(k'_1 \ln k'_1 - k'_1 + k'_2 \ln k'_2 - k'_2 \right) - \sum_n{}' \, \beta \frac{\sin\theta_n}{m+1} \frac{G_j(\phi_s)}{c(y_\nu)} \frac{dk'_{\nu n}}{d\eta} \ln k'_{\nu n} \right] - \sum_n{}' \, b_{\nu n}\, q_j(\eta_n)\, \pi I_{j\nu n} \right\} \tag{25}$$

where $b_{\nu\nu}$ and $b_{\nu n}$ are Multhopp's quadrature weighting factors and are defined in appendix B for convenience. The detailed expression for $dk'_{\nu n}/d\eta$ is also given in appendix B. Equation (25) represents the new system of equations for determining the values of the loading function q_j.

REGION OF VALIDITY FOR THE LSC

From the definition, it is known that

$$k'^2 = \frac{(A+B)^2 - 1}{4AB} = \frac{(1-X)^2 + X^2 + 2Y^2 - 1 + 2AB}{4\sqrt{(1-X)^2 + Y^2}\sqrt{X^2+Y^2}} = \frac{X^2 - X + Y^2 + \sqrt{(1-X)^2+Y^2}\sqrt{X^2+Y^2}}{2\sqrt{(1-X)^2+Y^2}\sqrt{X^2+Y^2}} \tag{26}$$

As indicated previously, the logarithmic-singularity correction term (LSC) must be removed before accurate spanwise integration across the second-order singularity can be performed. However, the LSC is only important if k' is small. In accordance with equation (17), small k' implies small Y, say $Y = O(\epsilon)$, where ϵ is a small positive number. To see how this requirement restricts the values of X, consider $X = O(\epsilon)$ also. Examination of equation (26) shows that

$$k'^2 = \frac{\epsilon\, C(1)}{\epsilon\, O(1)} = O(1)$$

It follows that the LSC is not important in this case. In fact, references 5 and 8 show that the correction term is algebraic, $O\left(|Y_0|^{1/2}\right)$, at the leading edge rather than logarithmic. When combined with the second-order singularity in equation (5), a singularity, $O\left(|Y_0|^{-3/2}\right)$, remains in the spanwise integration which must be performed with the finite-part concept. Since $X = O(\epsilon)$ implies that a control point is close to the wing leading edge, there must be a transition region near the leading edge in which the logarithmic singularity changes to an algebraic one. In this transition region, any use of the LSC would be improper, as it would misrepresent the type of singularity.

On the other hand, consider $Y = O(\epsilon)$, $X = O(1)$, and $(1 - X) = O(1)$. Then, by binomial expansion,

$$k'^2 = \frac{X^2 - X + O(\epsilon^2) + |X||1 - X|\left[1 + \frac{O(\epsilon^2)}{(1 - X)^2}\right]^{1/2}\left[1 + \frac{O(\epsilon^2)}{X^2}\right]^{1/2}}{2|X||1 - X|\left[1 + \frac{O(\epsilon^2)}{(1 - X)^2}\right]^{1/2}\left[1 + \frac{O(\epsilon^2)}{X^2}\right]^{1/2}}$$

$$= \frac{O(\epsilon^2)\left[1 + O(1) + O(\epsilon^2)\right]}{2|X||1 - X|\left[1 + O(\epsilon^2)\right]} \qquad (27)$$

where it is assumed that $0 < X < 1$. Therefore, k' is $O(\epsilon)$, and the LSC is proper in this case. In other words, to use the LSC properly, the chordwise control points should be located so that

$$|X| >> |Y| \qquad Y = O(\epsilon) \qquad (28a)$$

for control points near the leading edge and

$$|(1 - X)| >> |Y| \qquad Y = O(\epsilon) \tag{28b}$$

for control points near the trailing edge.

To analyze these conditions further, consider relation (28a). It is convenient to express both X and Y in terms of Y_0. Using the first two terms of a Taylor series and the binomial expansion leads to

$$X = \frac{1}{2}\frac{x - \xi_{le}(\eta)}{c(\eta)} = \frac{1}{2}\frac{x - \xi_{le}(y) - \xi'_{le}(y)\,(\eta - y)}{c(y)\left[1 + \frac{c'(y)}{c(y)}\,(\eta - y)\right]}$$

$$= \left(X_0 + \frac{\tan \Lambda_{le}}{\beta} Y_0\right)\left\{1 + \frac{c'(y)}{c(y)}\,(y - \eta) + \left[\frac{c'(y)}{c(y)}\right]^2 (y - \eta)^2\right\} + O\left(Y_0{}^3\right)$$

$$= X_0 + \frac{Y_0}{\beta}\left[(1 - X_0)\tan \Lambda_{le} + X_0 \tan \Lambda_{te}\right] + \frac{Y_0{}^2}{\beta^2}\left(\tan \Lambda_{te} - \tan \Lambda_{le}\right)\left[(1 - X_0)\tan \Lambda_{le}\right.$$

$$\left. + X_0 \tan \Lambda_{te}\right] + O\left(Y_0{}^3\right) \tag{29}$$

where

$$\left.\begin{aligned} &\frac{1}{2}\left(\tan \Lambda_{te} - \tan \Lambda_{le}\right) = \frac{dc(y)}{dy} \\ &\tan \Lambda_{le} = \frac{d\xi_{le}(y)}{dy} \qquad\qquad \tan \Lambda_{te} = \frac{d\xi_{te}(y)}{dy} \end{aligned}\right\} \tag{30}$$

and

$$Y_0 = \frac{\beta(y - \eta)}{2\,c(y)} \tag{31}$$

Similarly,

$$Y = \frac{1}{2}\beta\frac{y-\eta}{c(\eta)} = \frac{1}{2}\beta\frac{y-\eta}{c(y)}\left[1 + \frac{c'(y)}{c(y)}(y-\eta)\right] + O\left(Y_0^3\right)$$

$$= Y_0 + \frac{Y_0^2}{\beta}\left(\tan\Lambda_{te} - \tan\Lambda_{le}\right) + O\left(Y_0^3\right) \tag{32}$$

If terms of the order of Y_0^2 are neglected in equations (29) and (32), the condition for the validity of the LSC near the leading edge (eq. (28a)) becomes

$$\left|X_0 + \frac{Y_0}{\beta}\left[(1-X_0)\tan\Lambda_{le} + X_0\tan\Lambda_{te}\right]\right| >> |Y_0| \tag{33}$$

A critical test of relation (33) would be when Y_0 is negative, $|y-\eta|$ is as small as possible to ensure the importance of the LSC, such as in the tip region, and X_0 is for the control point nearest the leading edge. In numerical calculation, $|y-\eta|$ is small if the number of spanwise integration points, m, is reasonably large. Reasonably large m is necessary for accurate integration. Therefore, relation (33) provides a criterion to determine how close to the leading edge the control points are allowed to be located. Relation (33) shows that the condition involves the effects of aspect ratio and taper ratio (through c(y)), Mach number, and the sweep angle. For example, relation (33) shows that the larger the sweep angle and subsonic Mach number, the more difficult it is to satisfy the relation. Most existing kernel function methods require empirical relations between the numbers of chordwise and spanwise control points for convergence. Such relations are usually applicable only to a limited set of planforms and a limited range of flow conditions. Relation (33) offers a theoretical relation to ensure the correct representation of logarithmic singularity and therefore to ensure better accuracy of the numerical integration.

A similar criterion can also be set up for control points near the trailing edge. However, it is known that the behavior of I_j at the trailing edge in the spanwise integral is algebraic, being $O\left(|Y_0|^{3/2}\right)$, compared with $O\left(|Y_0|^{1/2}\right)$ at the leading edge (ref. 5). Therefore, near the trailing edge the singularity in the spanwise integration of equation (5) is at most $O\left(|Y_0|^{-1/2}\right)$ which is integrable, even if the behavior in the integrals I_j has not been accounted for at all. The situation is quite different at the leading edge. Hence, a similar criterion for control points near the trailing edge is less important than that of relation (33) in the overall accuracy of computation.

CRITERION FOR SELECTION OF N,m SET

In order to satisfy relation (33), reexpressing it in the following slightly different form is useful:

$$\left| X_0 + \frac{Y_0}{\beta} \tan \Lambda_{le} - \frac{Y_0}{\beta} X_0 \left(\tan \Lambda_{le} - \tan \Lambda_{te} \right) \right| >> \left| Y_0 \right| \tag{34}$$

Near the leading edge the critical test mentioned previously is for Y_0 becoming a small negative number. Thus, since Λ_{le} is in general larger than Λ_{te}, the smallest value of the left-hand side would occur if the third term were omitted because its contribution would be only a small positive number under these conditions. So, let

$$\left| X_0 + \frac{Y_0}{\beta} \tan \Lambda_{le} \right| >> \left| Y_0 \right| \tag{35}$$

The left-hand side of relation (35) must be positive without the absolute value signs, since it represents X. (See the discussion following eq. (27).) Therefore, the following relation must be satisfied:

$$X_0 > \frac{\left| Y_0 \right|}{\beta} \tan \Lambda_{le} \tag{36}$$

C_1 Dependence

In order to determine the magnitude of the terms required to satisfy the "much larger than" stipulations, it is expedient to reformulate relation (35) into a simple "larger than" test. This can be done by rewriting relation (35) as

$$X_0 + \frac{Y_0}{\beta} \tan \Lambda_{le} > C_1 \left| Y_0 \right| \tag{37}$$

with C_1 being an as yet undetermined constant. In order for X_0 to be as small as possible, it should be written for the first control point to give

$$X_0 = \frac{x - \xi_{le}(y)}{2\, c(y)} = \frac{1 - \cos\left(\frac{2\pi}{2N + 1} \right)}{2} \tag{38}$$

where N is the number of control points in the chordwise direction. Similarly, for Y_0 to be as small as possible, it should be written for the integrating station nearest the wing tip and the control point at the next inboard location:

$$Y_0 = \frac{\beta(y - \eta)}{2\ c(y)} = \frac{-\beta}{2\ c(y)}\left[\sin\frac{(m - 1)\pi}{2(m + 1)} - \sin\frac{(m - 3)\pi}{2(m + 1)}\right]\left(\frac{b}{2}\right) \tag{39}$$

where m is the number of spanwise integration stations from tip to tip. Thus, the problem of finding an N,m set reduces to determining a proper value for C_1. The approach taken is to select different values of C_1 and then examine the consequences in terms of N and m which satisfy relation (37) and yield ratios of near-field to far-field vortex drag $C_{D,ii}/C_{D,i}$ near unity. According to Multhopp (ref. 1), the drag ratio information provides an indication as to the accuracy of the solution. This ratio, which ideally would have a value of unity, is very sensitive to variations in N and m and is obtained from the equations given in reference 7. The results of this study are presented in table I for three thin flat wings at $M = 0$. Wing 1 is an aspect-ratio-7 rectangular wing, wing 2 is an aspect-ratio-3.5 sheared rectangular wing with $\Lambda = 45^{\circ}$, and wing 3 is an aspect-ratio-2 delta wing.

In order to determine a satisfactory N,m set, initial values are needed. The initial values selected for all three wings were $N = 8$ and $m = 23$. The sets which satisfy relation (37) are listed in table I. The solutions for wings 1 and 2 indicate that $C_1 \approx 10$ would be reasonable. It is also clear from table I that (1) though N and m may satisfy relation (37), there is no guarantee that the $C_{D,ii}/C_{D,i}$ value will be near 1; and (2) for wing 3, no satisfactory value of C_1 is determined. That wing 3 should present a convergence problem is perhaps not surprising, since the leading- and trailing-edge sweep angles of this configuration are markedly different, and thereby the term omitted in going from relation (34) to (35) is emphasized. (See the discussion which follows relation (34).) However, when relation (34) was used to determine the N,m set for wing 3 at $C_1 = 10$, no change in the set and consequently in $C_{D,ii}/C_{D,i}$ occurred. (See table I.) The difficulty seems to be in the initial estimates of N and m.

The slow convergence for wing 3 compared with that for the other wings is caused by a difference in Y_0. For the other wings Y_0 is $O(\epsilon)$, whereas for wing 3, Y_0 is $O(1)$. This change in the order of magnitude of Y_0 is caused by the diminishing chord at the tip of wing 3 and can be seen from an examination of equation (39). Note that from equation (38), X_0 is $O(1)$ for all wings. From the preceding discussion, it is apparent that for wing 3 the left-hand side of relation (37) is the difference between two $O(1)$ terms; thus, the relationship has less impact on a proper choice of N and m, or it becomes a

TABLE I.- EFFECT OF C_1 ON N, m, AND ON $C_{D,ii}/C_{D,i}$

[M = 0]

C_1	Wing 1			Wing 2			Wing 3		
	N	m	$C_{D,ii}/C_{D,i}$	N	m	$C_{D,ii}/C_{D,i}$	N	m	$C_{D,ii}/C_{D,i}$
2	7	49	0.5105	8	47	-0.3590	8	43	-1.5514
4	6	63	-.1250	7	55	.2571	7	47	-.8792
6	5	63	.8997	6	57	.7224	7	55	-.5417
8	5	73	.9700	6	65	.8687	6	51	-.1010
10	4	67	1.0746	5	59	1.0380	6	57	[b].1055
[a]10	4	67	1.0746	5	59	1.0380	5	63	.7597
12	4	75	1.0883	5	67	1.0957	6	63	.2743
14	4	79	1.0920	5	71	1.1109	6	65	.3230
[a]14	4	79	1.0920	5	71	1.1109	5	73	.8793
16	3	67	1.0573	5	77	1.1208	5	59	.6914
18	3	71	1.0558	5	79	1.1215	5	63	.7597
[a]18	3	71	1.0558	5	79	1.1215	4	67	.9717
20	3	75	1.0536	4	71	1.0758	5	65	.7889
22	3	79	1.0508	4	73	1.0681	5	67	.8153
[a]22	3	79	1.0508	4	73	1.0681	4	75	.9862
24	3	101	1.0346	4	75	1.0606	5	71	.8602

[a]Relation (40) was used in addition to relation (37).

[b]Unchanged when using relation (34) rather than relation (35).

poorer "necessary but not sufficient" condition. The net effect, as seen in table I, is generally to keep N larger and m smaller than N and m for the other wings, for a fixed value of C_1. This amounts to putting the first chordwise control point nearer the leading edge while slightly increasing the spanwise distance between tipmost integrated and integrating stations.

Another procedure was implemented which extended relation (36) to the following more stringent form:

$$X_0 > C_1 \frac{|Y_0|}{\beta} \tan \Lambda_{le} \tag{40}$$

This relation biases X_0 away from the leading edge by weighting the right-hand side of the inequality. Relation (40) is used in addition to the "necessary but not sufficient" relation (37). Relations (40) and (37) were used for the following values of C_1: 10, 14, 18, and 22; these results are identified in table I. Using relations (40) and (37) had no effect on the wing 1 and 2 solutions; however, there were improvements for wing 3. Note that the $C_{D,ii}/C_{D,i}$ values more quickly approach unity with a smaller product of N and m by using relationships (40) and (37) than with relationships (36) and (37).

Effect of Initial Values of N and m

Thus far the results have shown that (1) $C_1 \approx 10$ is a good compromise choice for untapered wings regardless of the relationships used to determine satisfactory N,m sets and (2) for tapered wings, relations (40) and (37) yield improved results. Consequently, a hypothesis is offered that the N,m sets which result from all preceding relationships may be dependent on the initial values of N and m. This dependence is suspected because even with local satisfaction of the relationships, there were $C_{D,ii}/C_{D,i}$ values in table I far from unity for ranges of C_1. It appears that both "local" and "global" conditions must be satisfied. Therefore, depending on the initial values of N and m, or where the procedure starts from, both conditions may not be satisfied.

To examine this hypothesis, a study was conducted based on relations (40) and (37) with $C_1 = 10$. Results of this study are summarized in table II for 20 sets of initial N and m values.

Before the results are analyzed, the computerized procedure for determining the N,m sets should be explained. It is as follows:

(1) The initial guesses for N and m are used in relation (40), in conjunction with equations (38) and (39); if the relation is not satisfied, m is increased by 4 and another attempt is made.

(2) Increasing m by 4 continues until relation (40) is satisfied or $m > 101$ in which case m is set to 101, N is reduced by 1, and the code attempts to satisfy relation (40) again.

(3) If this reduction in N is not successful in satisfying relation (40), then N is again reduced by 1, still with $m = 101$, and the process is repeated until the relation is satisfied or $N = 0$. If $N = 0$, the program writes a message and stops.

(4) Once relation (40) is satisfied, the N,m combination that led to satisfaction of relation (40) is used in relation (37). If either the N or m is not appropriate for relation (37), it is incremented as described previously, and the procedure returns to step (1) where relation (40) must once again be satisfied.

TABLE II.- EFFECT OF INITIAL N AND m VALUES

$[C_1 = 10;\ M = 0]$

Initial value		Wing 1			Wing 2			Wing 3		
N	m	N	m	$C_{D,ii}/C_{D,i}$	N	m	$C_{D,ii}/C_{D,i}$	N	m	$C_{D,ii}/C_{D,i}$
2	11	2	39	1.1108	2	31	1.0271	2	31	0.9341
2	31	2	39	1.1108	2	31	1.0271	2	31	.9341
2	51	2	51	1.0863	2	51	1.0057	2	51	.9401
2	71	2	71	1.0607	2	71	1.0119	2	71	1.0047
2	91	2	91	1.0501	2	91	1.0177	2	91	1.1185
4	11	4	67	1.0746	4	51	1.1188	4	51	.8840
4	31	4	67	1.0746	4	51	1.1188	4	51	.8840
4	51	4	67	1.0746	4	51	1.1188	4	51	.8840
4	71	4	71	1.0827	4	71	1.0758	4	71	.9804
4	91	4	91	1.0939	4	91	1.0097	4	91	.9858
6	11	4	67	1.0746	5	59	1.0380	5	63	.7597
6	31	4	67	1.0746	5	59	1.0380	5	63	.7597
6	51	4	67	1.0746	5	59	1.0380	5	63	.7597
6	71	4	67	1.0746	5	59	1.0380	5	63	.7597
6	91	4	67	1.0746	5	59	1.0380	5	63	.7597
3	11	4	67	1.0746	5	59	1.0380	5	63	.7597
8	31	4	67	1 0746	5	59	1.0380	5	63	.7597
8	51	4	67	1.0746	5	59	1.0380	5	63	.7597
8	71	4	67	1.0746	5	59	1.0380	5	63	.7597
8	91	4	67	1.0746	5	59	1.0380	5	63	.7597

(5) After both relations are satisfied, a test is made to determine whether the product of N and $(m + 1)/2$ is larger than 200, the maximum allowed. This product is the unique number of modal factors on the wing under either symmetrical or antisymmetrical loading conditions.

(6) If the product is less than or equal to 200, the aerodynamic solution is started. If it is greater than 200, N is reduced by 1, m is reduced to

$$2\left\{\text{Greatest whole integer}\left[\frac{0.4(m+1)}{2}\right]\right\} - 1$$

and the procedure returns to step (1) for another start.

Examination of table II prompts the following observations:

(1) A few initial N and m values satisfied both relations and became the final N,m sets.

(2) For N = 6 or 8 and any m, the same final N and m values for a particular wing were obtained.

(3) The best overall $C_{D,ii}/C_{D,i}$ results were obtained with initial values of N = 2 and m = 71, and second best with initial values of N = 4 and m = 91.

(4) The delta wing (wing 3) shows the most sensitivity in $C_{D,ii}/C_{D,i}$ to the initial N and m values.

(5) For N = 2, $C_{D,ii}/C_{D,i}$ convergence generally occurs as initial values of m increase up to 71 for all wings and up to 91 for wing 1. For N = 4, $C_{D,ii}/C_{D,i}$ convergence occurs as initial values of m increase up to 91 for wings 2 and 3, whereas a mild divergence occurs for wing 1.

Computer Requirements

An overlayed computer program which permits up to 200 flat-wing modal factors and m = 101 requires approximately 77000_8 words in central memory for the symmetric mode of operation and more in the antisymmetric mode. These computer requirements for a potential-flow aerodynamic solution were considered excessive; thus, an operational version was developed which permits up to 100 flat-wing modal factors, m = 41, and requires only 51000_8 words of central memory in either mode of operation. The consequences of using this computer program are examined before the results obtained with it are discussed in the remainder of this report.

It should be pointed out that the procedure for finding a final N,m set from the initial values is slightly different in the smaller program. The difference is primarily that a value of m considered "large enough" (usually 41) is specified initially, so that only reduction in N is permitted. In general, this is satisfactory as is seen subsequently.

The primary consequence of using the smaller program is that for some wings a value of N which is too small results from relations (40) and (37). This leads to the boundary condition being satisfied only at as few as one chordwise position. The only way that this situation could be remedied is to increase the value of m allowable, essentially to return to the original program discussed. The operational computer program, with modest computational requirements, is capable of generating acceptable $C_{D,ii}/C_{D,i}$ results for most wings, whereas the original computer program with larger

computational requirements would provide acceptable $C_{D,ii}/C_{D,1}$ results for an unknown number of additional wings. Thus, the smaller program was chosen.

Comparison With Previous N,m Set Criterion

Reference 7 presents an approximate formula for the relationship between m and N which can be used to obtain results that are considered converged. This relationship was developed for rectangular wings and is

$$m \approx (4 \text{ to } 5)\beta A \frac{N}{4} \tag{41}$$

where A is aspect ratio. It is of interest to compare the formulation of reference 7 with relation (33) for the same wings. For rectangular wings, relation (33) reduces to

$$X_0 >> |Y_0| \tag{42}$$

By using equations (38) and (39), relation (42) can be rewritten as

$$\frac{1 - \cos\left(\frac{2\pi}{2N+1}\right)}{2} > C_1 \frac{\beta}{2\,c(y)}\left(\frac{b}{2}\right)\left\{\sin\left[\frac{(m-1)\pi}{2(m+1)}\right] - \sin\left[\frac{(m-3)\pi}{2(m+1)}\right]\right\} \tag{43}$$

which with reduction becomes

$$1 - \cos\left(\frac{2\pi}{2N+1}\right) > C_1\,\beta A\left[\cos\left(\frac{\pi}{m+1}\right) - \cos\left(\frac{2\pi}{m+1}\right)\right] \tag{44}$$

where the aspect ratio A is

$$A = \frac{b}{2\,c(y)} = \frac{b}{c_{av}}$$

For $\left[j\pi/(m+1)\right] < 1$, the right-hand side of relation (44) can be expanded to yield

$$1 - \cos\left(\frac{2\pi}{2N+1}\right) > C_1\,\beta A\left[\frac{3\pi^2}{2(m+1)^2} - \cdots\right] \tag{45}$$

and so,

$$m > \sqrt{\frac{C_1 \beta A 3\pi^2}{2\left[1 - \cos\left(\frac{2\pi}{2N+1}\right)\right]}} - 1 \qquad (46)$$

As an example of the different results obtained from these two procedures, consider the following values for the variables: $\beta = 1$, $A = 2$, $N = 4$, and $C_1 = 10$. Then equation (41) yields

$$m \approx 10$$

and equation (46) yields

$$m > 34$$

ADDITIONAL NUMERICAL STUDIES

Two additional numerical studies were made. The first examines the effect of the new LSC on the ratio of near-field to far-field vortex drag, and the second examines the effect of C_1 on the overall longitudinal aerodynamic results.

Effect of LSC on Vortex Drag Ratio

To show the improvement obtained by using the new LSC, the vortex drag ratios for two planforms, rectangular and delta, are studied over a range of N and m. Results using the LSC described in reference 7 are compared with those using the new LSC. It should be stated at the outset that for either LSC, vortex drag ratios near 1 are not to be expected for any arbitrary combination of N and m. Figure 1 presents the results of this numerical experimentation. The general conclusions for both planforms are that the new LSC (1) provides a stable solution which converges for a fixed value of N with increasing m and (2) yields convergence with increasing N only for a sufficiently large value of m.

The second conclusion is strongly related to relation (37) and signifies how violations of the N,m criterion decrease the solution accuracy. For example, if N is too small in relation to m, the downwash equation cannot be satisfied more accurately in the chordwise direction. If N is large, the misrepresentation of the logarithmic singularity near the leading edge reduces the accuracy, depending on the seriousness of the violation.

On the other hand, larger N can better satisfy the boundary condition so that the accuracy is increased. The whole problem is the following: N and m should be reasonably large so that the boundary condition can be better satisfied, while simultaneously the correct representation of the logarithmic singularity should be preserved. The N,m criterion (relation (37)) deals with the latter condition only. Table II illustrates the point that solutions for small N may be as good or better than those for larger N where each has an m value satisfying relations (37) and (40).

Effect of C_1 on Longitudinal Aerodynamic Results

Results of the second numerical study are presented in table III. In it the aerodynamic results obtained by employing both the old LSC from reference 7 and the new LSC for a variety of planforms are given. Reference 7 used the N,m relationship given by equation (41). Results using the new LSC are presented for two C_1 values, and relations (40) and (37) were employed for the N,m relationship.

Upon comparing the results in table III, the following observations can be made:

(1) The new values of x_{ac} are generally greater than those of reference 7.

(2) The C_{L_α} values vary only slightly with solution or C_1.

(3) The $C_1 = 10$ solutions generally have N values greater than the $C_1 = 14$ solutions.

(4) The $C_{D,ii}/C_{D,i}$ values for $C_1 = 10$ are often slightly farther from 1 than those for $C_1 = 14$, with the exception of the more complex planforms.

(5) At N,m combinations based on equation (41) the drag ratios for the new LSC solutions are generally farther from 1 than those obtained with the N,m combinations based on relations (40) and (37).

These observations indicate that the $C_1 = 10$ solutions are almost as good as those for $C_1 = 14$ and generally provide one more chordwise point for local downwash satisfaction. Hence, $C_1 = 10$ is used subsequently.

It should be noted again at this point that with m increased the new version would produce improved results. This is not necessarily true with the old version, as has been shown in figure 1.

TABLE III.- LONGITUDINAL AERODYNAMIC RESULTS USING OLD AND NEW LSC FOR SEVERAL PLANFORMS

Wing type	Aspect ratio	Λ, deg	λ	M	C_l	N	m	C_{L_α}		x_{ac}		$C_{D,ii}/C_{D,i}$		$C_{D,ii}/C_{D,i}$ using new LSC and equation (41)
								Old LSC (a)	New LSC	Old LSC (a)	New LSC	Old LSC (a)	New LSC	
Rectangular	2	0	1.0	0		4	11	0.0432		0.208		0.9943		0.9549
					10	4	41		0.0432		0.210		1.0337	
					14	3	41		.0431		.210		1.0288	
Rectangular	7	0	1.0	0		4	37	0.0768		0.239		1.0277		0.8414
					10	2	41		0.0777		0.244		1.1067	
					14	1	41		.0764		.250		1.2209	
Sweptback	2	45	1.0	0		4	11	0.0398		0.170		0.5211		0.5371
					10	4	41		0.0398		0.171		1.0610	
					14	3	41		.0396		.169		1.0297	
Sweptback	6	46.17	0.60	0		4	31	0.0615		0.252		-0.5252		-0.5573
					10	2	41		0.0607		0.268		1.0818	
					14	1	41		.0596		.263		1.0363	
Delta	2	63.4	0	0.13		4	11	0.0391		0.363		-0.8669		-1.5331
					10	3	41		0.0385		0.390		0.9721	
					14	2	41		.0385		.389		.9372	
Cropped diamond	2	45	0.25	0		4	9	0.0432		0.248		0.5011		0.3748
					10	4	41		0.0430		0.266		1.0749	
					14	3	41		.0430		.265		1.0009	
Backward, cropped delta	2	0	0.25	0		4	11	0.0426		0.191		0.9567		0.9813
					10	4	41		0.0424		0.193		1.0325	
					14	3	41		.0423		.191		1.0355	
Double delta	1.415	82/60	0.10	0		4	9	0.0317		0.433		-0.4254		-0.8222
					10	4	41		0.0313		0.459		1.1348	
					14	3	41		.0313		.461		1.0706	
Double delta	1.7	75/60	0.10	0		6	13	0.0365		0.375		-0.9101		-1.8484
					10	3	41		0.0362		0.388		1.0462	
					14	3	41		.0362		.388		1.0462	
Double delta	1.97	45/60	0.10	0		6	15	0.0407		0.324		-0.2372		-1.2064
					10	3	41		0.0404		0.334		1.0735	
					14	2	41		.0405		.335		.9196	

[a]Taken from reference 7.

APPLICATIONS

Table IV presents the vortex drag ratios for a wide variety of planforms and subsonic Mach numbers. All results are obtained from the hands-off computer solution,

TABLE IV.- VORTEX DRAG RATIOS FOR A VARIETY OF PLANFORMS AND MACH NUMBER

$[C_1 = 10]$

Wing type	Aspect ratio	Λ, deg	λ	M	N	m	$C_{D,ii}/C_{D,i}$
Rectangular	0.2	0	1.0	0	8	23	0.9799
Rectangular	.3	0	1.0	0	7	23	1.0956
Rectangular	.4	0	1.0	0	6	23	.9721
Rectangular	1.0	0	1.0	0	5	31	1.0344
Rectangular	2.0	0	1.0	0	3	31	1.0350
Rectangular	3.0	0	1.0	0	3	41	1.0329
Rectangular	7.0	0	1.0	0	2	41	1.1067
Sheared rectangular	3.5	0	1.0	.3	3	41	1.0366
Sheared rectangular	3.5	20	1.0	.3	3	41	1.0503
Sheared rectangular	3.5	40	1.0	.3	3	41	1.0734
Sheared rectangular	3.5	50	1.0	.3	3	41	1.1348
Sheared rectangular	3.5	60	1.0	.3	2	41	.9751
Sheared rectangular	3.5	70	1.0	.3	1	41	1.1589
Sheared rectangular	3.5	75	1.0	.3	1	41	1.1241
Sweptback	1.0	45	1.0	0	4	41	1.0179
Sweptback	2.0	45	1.0	0	4	41	1.0610
Delta	4.0	45	0	.6	3	41	.9705
Cropped delta	3.273	45	.1	.6	3	41	1.0764
Cropped delta	2.667	45	.2	.6	4	41	1.1397
Cropped delta	2.154	45	.3	.6	4	41	1.0648
Cropped delta	1.714	45	.4	.6	4	41	1.0279
Cropped delta	1.333	45	.5	.6	4	41	1.0197
Cropped delta	1.668	63	.1	.6	3	41	1.0378
Cropped delta	1.359	63	.2	.6	4	41	1.0986
Cropped delta	1.097	63	.3	.6	4	41	1.0207
Cropped delta	.873	63	.4	.6	4	41	.9887
Cropped delta	.873	63	.4	0	4	41	1.0313
Cropped arrow	1.069	63	.54	0	4	41	1.0148
Cropped arrow	1.917	63	.29	0	3	41	1.0165
Cropped diamond	.738	63	.32	0	4	41	1.0248
Arrow	3.25	60	0	0	2	41	.9397
Diamond	1.75	60	0	0	3	41	1.0379

with the N,m sets derived from relations (40) and (37) and equations (38) and (39) with $C_1 = 10$. Many of the initial N and m values used in the solutions were $N = 4$ and $m = 41$. From table IV, it can be seen that generally the near-field drag is within ±10 percent of the far-field drag, and in many cases the error is even smaller, within ±5 percent. The larger errors do not occur in any systematic pattern that could be associated with a particular wing type, aspect ratio, sweep, taper ratio, or Mach number, nor do the values of N and m selected by the program lead to any correlative trends. It is expected that these errors can be reduced if m was increased beyond the 41 limit imposed by the smaller computer code.

As noted in the discussion of relation (33), the computer program may have difficulty in satisfying that relation for wings with higher leading-edge sweep. An example of this occurs for the sheared wing results given in table IV. A solution for $\Lambda = 80^{\circ}$ was attempted but the conditions in relation (33) could not be satisfied for $m = 41$ or $m < 41$ at any N value. Hence, it was not included. This led to the use of $\Lambda = 75^{\circ}$ which was not too large to satisfy the aspect ratio, taper ratio, and Mach number requirements inherent in relation (33).

CONCLUDING REMARKS

This report has presented the development of a new logarithmic-singularity correction term to be used in Multhopp lifting-surface computer programs. One novel aspect of the correction term is that it is expressed as a function of the complementary modulus of the complete elliptic integrals. In this form the correction term contains both chordwise and spanwise variations in the integrand near the spanwise singularity. It also leads to the establishment of a relationship between the chordwise control point nearest the wing leading edge and the spanwise distance between integrating stations and control points, so that the correction term computed will be valid. With this term set, reliable aerodynamic results can be obtained in a single computer pass. In addition, even if the previously mentioned relationship is not used, it has been determined that the new correction term leads to stable converging solutions for a fixed number of chordwise control points with increasing number of spanwise integration stations. Stable converged solutions have not always occurred for arbitrary planforms and subsonic Mach numbers.

Numerical studies indicate that with this new relationship, the ratios of near-field to far-field vortex drag for a variety of planforms and subsonic Mach numbers are generally between 0.90 and 1.10 and in many instances between 0.95 and 1.05.

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APPENDIX A

EVALUATION OF J_1, J_2, AND $dk'/d\eta$

Evaluation of J_1

The closed-form expression for J_1 has been obtained by Wegener and published in reference 10. However, details of the derivation are not available. They are included here for the sake of completeness. In this appendix, all page numbers refer to reference 11.

J_1 can be written as

$$J_1 = \int_0^1 \sqrt{\frac{1 - X_1}{X_1}} \left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}} \right] dX_1$$

$$= \int_0^1 \sqrt{\frac{1 - X_1}{X_1}}\, dX_1 + \int_0^1 \frac{(1 - X_1)(X - X_1)}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}\, dX_1$$

$$= \int_0^1 \sqrt{\frac{1 - X_1}{X_1}}\, dX_1 + X \int_0^1 \frac{dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$- (1 + X) \int_0^1 \frac{X_1\, dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}} + \int_0^1 \frac{X_1^2\, dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$= J_{11} + J_{12} + J_{13} + J_{14} \tag{A1}$$

From p. 133, some parameters involved in the results are defined as follows:

$$A^2 = (1 - X)^2 + Y^2 \tag{A2a}$$

$$B^2 = X^2 + Y^2 \tag{A2b}$$

$$k^2 = \frac{1 - (A - B)^2}{4AB} \tag{A2c}$$

$$g = \frac{1}{\sqrt{AB}} \tag{A2d}$$

$$\varphi = \pi \tag{A2e}$$

The first integral in equation (A1) J_{11} can be shown to be $\pi/2$. Now, by Item 259.00 on p. 133,

$$J_{12} = X \int_0^1 \frac{dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$= Xg\ F(\varphi,k) = 2Xg\ K(k) \tag{A3}$$

Again, from Item 259.03 on p. 133, it is found that

$$J_{13} = -(1 + X) \int_0^1 \frac{X_1\, dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$= -g(1 + X)\left[-\frac{2B}{A - B} K(k) + \frac{A + B}{A - B} \Pi\left(\frac{\alpha^2}{\alpha^2 - 1},k\right)\right] \tag{A4}$$

where f_1, defined in Item 361.54 on p. 215, equals zero and

$$\left.\begin{aligned} \alpha_2 &= -1 \\ \alpha &= \frac{A - B}{A + B} \\ 1 - \alpha^2 &= \frac{4AB}{(A + B)^2} \\ \frac{\alpha^2}{\alpha^2 - 1} &= -\frac{(A - B)^2}{4AB} \end{aligned}\right\} \tag{A5}$$

Similarly,

$$J_{14} = \int_0^1 \frac{X_1^2\, dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$= \frac{gB^2}{(A - B)^2}\left[\alpha_2^2\, F(\varphi,k) + 2\alpha_2(\alpha - \alpha_2)\frac{1}{1 - \alpha^2}\,\Pi\left(\varphi,\frac{\alpha^2}{\alpha^2 - 1},k\right) + (\alpha - \alpha_2)^2 R_2\right]$$

where from Item 341.04 on p. 206,

$$R_2 = \frac{1}{(\alpha^2 - 1)(k^2 + \alpha^2 k'^2)}\left\{\left[\alpha^2(2k^2 - 1) - 2k^2\right]R_1 + 2k^2 R_{-1} - k^2 R_{-2} + \frac{\alpha^3\, \mathrm{sn}\, u\, \mathrm{dn}\, u}{1 + \alpha\, \mathrm{cn}\, u}\right\}\Bigg|_{\varphi=\pi}$$

where $\mathrm{cn}\, u = \cos\varphi$, $\mathrm{sn}\, u = \sin\varphi$, $\mathrm{dn}\, u = \sqrt{1 - k^2\sin^2\varphi}$, and

$$R_1 = \frac{1}{1 - \alpha^2}\,\Pi\left(\varphi,\frac{\alpha^2}{\alpha^2 - 1},k\right)$$

$$R_{-1} = F(\varphi,k)$$

$$R_{-2} = \frac{1}{k^2}\left[\left(k^2 - \alpha^2 k'^2\right) F(\varphi,k) + \alpha^2\, E(\varphi,k)\right]$$

Hence,

$$J_{14} = \frac{gB^2}{(A - B)^2}\left\{\alpha_2^2\, F(\varphi,k) + \frac{2\alpha_2(\alpha - \alpha_2)}{1 - \alpha^2}\,\Pi\left(\varphi,\frac{\alpha^2}{\alpha^2 - 1},k\right)\right.$$

$$+ \frac{(\alpha - \alpha_2)^2}{(\alpha^2 - 1)(k^2 + \alpha^2 k'^2)}\left[\frac{\alpha^2(2k^2 - 1) - 2k^2}{1 - \alpha^2}\,\Pi\left(\varphi,\frac{\alpha^2}{\alpha^2 - 1},k\right)\right.$$

$$\left.\left. + 2k^2\, F(\varphi,k) - \left(k^2 - \alpha^2 k'^2\right) F(\varphi,k) - \alpha^2\, E(\varphi,k)\right]\right\}$$

APPENDIX A

By substituting equations (A2) and (A5), J_{14} becomes

$$J_{14} = g\left\{-\frac{2B}{A-B}K(k) + 2AB\,E(k) + \frac{1}{2}\frac{A+B}{A-B}\left[2 - \left(A^2 - B^2\right)\right]\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right)\right\} \tag{A6}$$

Combining the expressions for J_{11}, J_{12}, J_{13}, and J_{14} (eqs. (A3), (A4), and (A6)) gives

$$J_1 = \frac{\pi}{2} + 2Xg\,K(k) - g(1+X)\left[-\frac{2B}{A-B}K(k) + \frac{A+B}{A-B}\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right)\right]$$

$$+ g\left[-\frac{2B}{A-B}K(k) + 2AB\,E(k) + \frac{A+B}{A-B}\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right) - \frac{1}{2}(A+B)^2\,\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right)\right]$$

$$= \frac{\pi}{2} + 2g\left[\frac{XA}{A-B}K(k) + AB\,E(k) - \frac{1}{2}\frac{A+B}{A-B}X\,\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right) - \frac{1}{4}(A+B)^2\,\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right)\right]$$

From Item 410.01 on p. 225, $\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right)$ can be written as

$$\Pi\left(\frac{\alpha^2}{\alpha^2-1},k\right) = 4ABk^2\,K(k) + \frac{\pi\frac{(A-B)^2}{4AB}\Lambda_0(\psi',k)}{2\sqrt{\frac{(A-B)^2}{4AB}\frac{(A+B)^2}{4AB}\frac{1}{4A}}}$$

$$= 4ABk^2\,K(k) + \frac{\pi\sqrt{AB}\,|A-B|}{|A+B|}\Lambda_0(\psi',k) \tag{A7}$$

where

$$\psi' = \sin^{-1}|A-B|$$

By equation (A7), J_1 becomes

$$J_1 = \frac{\pi}{2} + 2g\left\{\frac{XA}{A-B}K(k) + AB\,E(k) - \frac{1}{2}\frac{A+B}{A-B}X\left[4ABk^2\,K(k) + \pi\sqrt{AB}\left|\frac{A-B}{A+B}\right|\Lambda_0(\psi',k)\right]\right.$$

$$\left. - \frac{1}{4}(A+B)^2\left[4ABk^2\,K(k) + \pi\sqrt{AB}\left|\frac{A-B}{A+B}\right|\Lambda_0(\psi',k)\right]\right\}$$

$$= \frac{\pi}{2} + 2g\left[\frac{XA}{A-B}K(k) + AB\,E(k) - \frac{AB(A+B)}{A-B}k^2\,K(k) - \frac{\pi\sqrt{AB}|A-B|}{4(A-B)}\Lambda_0(\psi',k)\right] \qquad (A8)$$

It is known (see p. 36) that for any ψ',

$$\Lambda_0(-\psi',k) = -\Lambda_0(\psi',k)$$

Hence,

$$\frac{|A-B|}{A-B}\Lambda_0(\psi',k) = \Lambda_0(\psi,k)$$

where

$$\psi = \sin^{-1}(A-B) \qquad (A9)$$

The coefficient of $K(k)$ in equation (A8) can be simplified as follows:

$$\frac{XA}{A-B} - \frac{AB(A+B)}{A-B}k^2 = \frac{XA}{A-B} - \frac{AB(A+B)}{A-B}\frac{1-(A-B)^2}{4AB}$$

$$= \frac{2A\left[1 - \left(A^2 - B^2\right)\right] - (A+B) + \left(A^2 - B^2\right)(A-B)}{4(A-B)}$$

$$= \frac{-A\left(A^2 - B^2\right) - B\left(A^2 - B^2\right) + A - B}{4(A-B)}$$

$$= \frac{-(A+B)^2 + 1}{4} = -ABk'^2$$

Thus, substituting for g from equation (A2) reduces J_1 to

$$J_1 = \frac{\pi}{2} + 2\sqrt{AB}\left[E(k) - k'^2\,K(k)\right] - \frac{\pi}{2}\,\Lambda_0(\psi,k)$$

$$= \frac{\pi}{2}\left\{1 - \Lambda_0(\psi,k) + \frac{4}{\pi}\sqrt{AB}\left[E(k) - k'^2\,K(k)\right]\right\} \qquad \text{(A10)}$$

Evaluation of J_2

J_2 can be written as

$$J_2 = \int_0^1 \frac{1}{\sqrt{X_1(1 - X_1)}}\left[1 + \frac{X - X_1}{\sqrt{(X - X_1)^2 + Y^2}}\right] dX_1$$

$$= \int_0^1 \frac{dX_1}{\sqrt{X_1(1 - X_1)}} + X\int_0^1 \frac{dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

$$- \int_0^1 \frac{X_1\,dX_1}{\sqrt{X_1(1 - X_1)}\sqrt{(X - X_1)^2 + Y^2}}$$

The first integral in the expression for J_2 can be shown to be π. The last two integrals can be expressed in the same way that J_{12} and J_{13} were (eqs. (A3) and (A4)). The resulting expression for J_2 is

$$J_2 = \pi + 2Xg\,K(k) - g\left[-\frac{2B}{A - B}\,K(k) + \frac{A + B}{A - B}\,\Pi\left(\frac{\alpha^2}{\alpha^2 - 1},k\right)\right]$$

Using equation (A7) for $\Pi\left(\frac{\alpha^2}{\alpha^2 - 1},k\right)$ results in

$$J_2 = \pi + 2Xg\,K(k) - g\left\{-\frac{2B}{A - B}\,K(k) + \frac{A + B}{A - B}\left[4ABk^2\,K(k) + \pi\sqrt{AB}\,\frac{|A - B|}{|A + B|}\,\Lambda_0(\psi',k)\right]\right\}$$

APPENDIX A

The coefficient of $K(k)$ can be reduced to

$$2g\left[X + \frac{B}{A - B} - \frac{2ABk^2(A + B)}{A - B}\right] = 2g\left[\frac{2XA - 2XB + 2B - (A + B) + \left(A^2 - B^2\right)(A - B)}{2(A - B)}\right]$$

$$= 2g\left[\frac{2XA - 2XB - 2B - A - B + (1 - 2X)(A - B)}{2(A - B)}\right]$$

$$= 0$$

It follows that

$$J_2 = \pi\left[1 - \Lambda_0(\psi,k)\right] \tag{A11}$$

where ψ is defined in equation (A9).

Evaluation of $dk'/d\eta$

For $\eta \to y$ $\left(\text{i.e., } X \approx X_0\right)$, the following relation can be obtained from the definitions of X and ϕ:

$$\frac{1}{2}(1 - \cos\phi) = \frac{x - \xi_{le}(y)}{2\,c(y)}$$

Hence, x is related to ϕ through

$$x = \xi_{le}(y) + c(y)(1 - \cos\phi) \tag{A12}$$

Thus, from the definitions of X and Y,

$$\left.\begin{aligned} X &= \frac{1}{2}\,\frac{\xi_{le}(y) + c(y)(1 - \cos\phi) - \xi_{le}(\eta)}{c(\eta)} \\ Y &= \frac{1}{2}\,\frac{\beta(y - \eta)}{c(\eta)} \end{aligned}\right\} \tag{A13}$$

APPENDIX A

Differentiation of equation (A13) with respect to η gives

$$\left.\begin{aligned}\frac{dX}{d\eta} &= \frac{1}{2}\left\{-\frac{\xi'_{le}(\eta)}{c(\eta)} - \frac{\left[\xi_{le}(y) + c(y)(1 - \cos\phi) - \xi_{le}(\eta)\right] c'(\eta)}{c^2(\eta)}\right\} \\ \frac{dY}{d\eta} &= \frac{\beta}{2}\left[-\frac{1}{c(\eta)} - \frac{(y - \eta)\, c'(\eta)}{c^2(\eta)}\right]\end{aligned}\right\} \qquad \text{(A14)}$$

where $c'(\eta)$ and $\xi'_{le}(\eta)$ can be expressed in terms of the sweep angles as follows:

$$\left.\begin{aligned}c'(\eta) &= \frac{1}{2}\left[\tan\Lambda_{te}(\eta) - \tan\Lambda_{le}(\eta)\right] \\ \xi'_{le}(\eta) &= \tan\Lambda_{le}(\eta)\end{aligned}\right\} \qquad \text{(A15)}$$

From definition,

$$k'^2 = 1 - k^2 = \frac{(A + B)^2 - 1}{4AB}$$

$$= \frac{X^2 - X + Y^2}{2AB} + \frac{1}{2} \qquad \text{(A16)}$$

Now, differentiation of equation (A16) gives

$$2k' \frac{dk'}{d\eta} = \frac{2X \frac{dX}{d\eta} - \frac{dX}{d\eta} + 2Y \frac{dY}{d\eta}}{2AB} - \frac{\left(X^2 - X + Y^2\right)}{2A^2B^2}\left(B \frac{dA}{d\eta} + A \frac{dB}{d\eta}\right)$$

But from the definitions of A and B

$$2A \frac{dA}{d\eta} = -2(1 - X) \frac{dX}{d\eta} + 2Y \frac{dY}{d\eta}$$

$$2B \frac{dB}{d\eta} = 2X \frac{dX}{d\eta} + 2Y \frac{dY}{d\eta}$$

and

$$B\frac{dA}{d\eta} + A\frac{dB}{d\eta} = \frac{1}{AB}\left(B^2A\frac{dA}{d\eta} + A^2B\frac{dB}{d\eta}\right)$$

$$= \frac{1}{AB}\left[-B^2(1 - X)\frac{dX}{d\eta} + B^2Y\frac{dY}{d\eta} + A^2X\frac{dX}{d\eta} + A^2Y\frac{dY}{d\eta}\right]$$

$$= \frac{1}{AB}\left[\left(A^2 + B^2\right)X - B^2\right]\frac{dX}{d\eta} + \frac{A^2 + B^2}{AB}Y\frac{dY}{d\eta}$$

Also,

$$X^2 - X + Y^2 = AB\left(2k'^2 - 1\right)$$

By these relations, it follows that

$$2k'\frac{dk'}{d\eta} = \frac{2X - 1}{2AB}\frac{dX}{d\eta} + \frac{Y}{AB}\frac{dY}{d\eta}$$

$$- \frac{AB\left(2k'^2 - 1\right)}{2A^2B^2}\left\{\frac{1}{AB}\left[\left(A^2 + B^2\right)X - B^2\right]\frac{dX}{d\eta} + \frac{A^2 + B^2}{AB}Y\frac{dY}{d\eta}\right\} \qquad \text{(A17)}$$

The coefficient of $dX/d\eta$ in equation (A17) can be written as

$$\frac{2X - 1}{2AB} - \frac{\left(2k'^2 - 1\right)\left[\left(A^2 + B^2\right)X - B^2\right]}{2A^2B^2}$$

$$= \frac{1}{2A^2B^2}\left[X\left(2AB + A^2 + B^2\right) - AB - B^2 - 2k'^2\left(A^2 + B^2\right)X + 2k'^2B^2\right]$$

$$= \frac{1}{2A^2B^2}\left[X\left(4ABk'^2 + 1\right) - AB - B^2 - 2k'^2\left(A^2 + B^2\right)X + 2k'^2B^2\right]$$

$$= \frac{1}{2A^2B^2}\left[-2Xk'^2(A - B)^2 + 2B^2k'^2 - 2ABk'^2\right]$$

$$= \frac{k'^2}{A^2B^2}\left[-X(A - B)^2 + B^2 - AB\right]$$

Similarly, the coefficient of $Y\frac{dY}{d\eta}$ in equation (A17) can be written as

$$\frac{1}{AB} - \frac{\left(2k'^2 - 1\right)\left(A^2 + B^2\right)}{2A^2B^2} = \frac{1}{2A^2B^2}\left[(A + B)^2 - 2k'^2\left(A^2 + B^2\right)\right]$$

$$= \frac{1}{2A^2B^2}\left[1 - 2k'^2(A - B)^2\right]$$

Substitution of these results into equation (A17) gives

$$2k'\frac{dk'}{d\eta} = \frac{k'^2}{A^2B^2}\left[-X(A - B)^2 + B^2 - AB\right]\frac{dX}{d\eta}$$

$$+ \frac{1}{2A^2B^2}\left[1 - 2k'^2(A - B)^2\right]\frac{\beta(\gamma - \eta)}{2\,c(\eta)}\frac{dY}{d\eta} \qquad \text{(A18)}$$

APPENDIX B

DERIVATION OF LSC

Before making any expansion of J_1 and J_2 for small Y, it is important to know the relation between Y^2 and k'^2 as defined in equation (A16). From equation (A16),

$$k'^2 = \frac{X^2 - X + Y^2}{2\sqrt{(1 - X)^2 + Y^2}\sqrt{X^2 + Y^2}} + \frac{1}{2}$$

Note that k'^2 approaches zero when Y vanishes. This expression can be reduced to a quadratic equation in Y^2 as follows:

$$4k'^2\left(k'^2 - 1\right) Y^4 + \left\{4k'^2\left(k'^2 - 1\right)\left[X^2 + (1 - X)^2\right] + 1\right\} Y^2$$

$$+ 4k'^2\left(k'^2 - 1\right) X^2(1 - X)^2 = 0 \qquad \text{(B1)}$$

This algebraic equation can be solved for Y^2 exactly and then the results expanded for small k'^2 or it can be solved by the perturbation method by noting that both Y^2 and k'^2 are small. By using the latter method, it is observed that to the first approximation,

$$Y^2 = 4k'^2X^2(1 - X)^2$$

Since the first term in equation (B1) is $O\left(k'^6\right)$, it may be neglected. Thus, by retaining terms involving k'^4,

$$Y^2 = -\frac{4k'^2\left(k'^2 - 1\right) X^2(1 - X)^2}{1 + 4k'^2\left(k'^2 - 1\right)\left[X^2 + (1 - X)^2\right]} + O\left(k'^6\right)$$

$$= 4k'^2X^2(1 - X)^2\left\{1 + k'^2\left[3 - 8X(1 - X)\right]\right\} + O\left(k'^6\right) \qquad \text{(B2)}$$

From equation (B2), it is seen that small k' implies always small Y, but the converse is not true. Hence, it is more natural to develop J_1 and J_2 for small k' rather than for small Y.

APPENDIX B

It is known that (see p. 36 of ref. 11)

$$\Lambda_0(\psi,k) = \frac{2}{\pi}\left[E(k)\ F(\psi,k') + K(k)\ E(\psi,k') - K(k)\ F(\psi,k')\right] \tag{B3}$$

From pp. 299-300 of reference 11, the following expansions for small k' are valid:

$$\left.\begin{aligned} E(k) &= 1 + \frac{k'^2}{2}\ln\frac{4}{k'} + \ldots \\ K(k) &= \ln\frac{4}{k'}\left(1 + \frac{k'^2}{4} + \ldots\right) + \ldots \\ F(\psi,k') &= \psi + \frac{1}{4}k'^2(\psi - \sin\psi\cos\psi) + \ldots \\ E(\psi,k') &= \psi - \frac{1}{4}k'^2(\psi - \sin\psi\cos\psi) + \ldots \end{aligned}\right\} \tag{B4}$$

where only the logarithmic components of $E(k)$ and $K(k)$ have been retained. Substitution of equation (B4) into equation (B3) gives

$$\Lambda_0(\psi,k) = \frac{2}{\pi}\left\{\left(1 + \frac{k'^2}{2}\ln\frac{4}{k'}\right)\left[\psi + \frac{k'^2}{4}(\psi - \sin\psi\cos\psi)\right]\right.$$

$$\left. + \ln\frac{4}{k'}\left(1 + \frac{k'^2}{4}\right)\left[\psi - \frac{k'^2}{4}(\psi - \sin\psi\cos\psi) - \psi - \frac{k'^2}{4}(\psi - \sin\psi\cos\psi)\right]\right\}$$

Therefore,

$$\begin{matrix}\text{Logarithmic} \\ \text{component} \\ \text{of } \Lambda_0\end{matrix} = \frac{2}{\pi}\left(-\sin\psi\cos\psi\,\frac{k'^2}{2}\ln k'\right) + O\left(k'^4\ln k'\right)$$

From equation (A9), $\sin\psi = A - B$, so that

$$\begin{matrix}\text{Logarithmic} \\ \text{component} \\ \text{of } \Lambda_0\end{matrix} = -\frac{1}{\pi}(A - B)\sqrt{1 - (A - B)^2}\,k'^2\ln k' + O\left(k'^4\ln k'\right) \tag{B5}$$

APPENDIX B

It follows that (see eq. (16))

$$\begin{matrix}\text{Logarithmic}\\ \text{component}\\ \text{of } J_2\end{matrix} = (A - B)\sqrt{1 - (A - B)^2}\, k'^2 \ln k' + O\left(k'^4 \ln k'\right) \tag{B6}$$

Similarly, by using equation (B4), equation (15) becomes

$$J_1 = \frac{\pi}{2}\left[1 - \Lambda_0(\psi,k) + \frac{4}{\pi}\sqrt{AB}\left(1 + \frac{k'^2}{2}\ln\frac{4}{k'} - k'^2 \ln\frac{4}{k'}\right)\right]$$

Then,

$$\begin{matrix}\text{Logarithmic}\\ \text{component}\\ \text{of } J_1\end{matrix} = \frac{\pi}{2}\left[\frac{1}{\pi}(A - B)\sqrt{1 - (A - B)^2}\, k'^2 \ln k' - \frac{2}{\pi}\sqrt{AB}\, k'^2 \ln\frac{4}{k'}\right] + O\left(k'^4 \ln k'\right)$$

$$= \frac{1}{2}\left[(A - B)\sqrt{1 - (A - B)^2} + 2\sqrt{AB}\right]k'^2 \ln k' + O\left(k'^4 \ln k'\right) \tag{B7}$$

Equations (B6) and (B7) can be further expanded by noting that

$$\left.\begin{aligned} A &= \left[(1 - X)^2 + Y^2\right]^{1/2} = \left[(1 - X)^2 + 4k'^2X^2(1 - X)^2 + O\left(k'^4\right)\right]^{1/2}\\ &= 1 - X + 2k'^2X^2(1 - X) + O\left(k'^4\right)\\ B &= \left[X^2 + Y^2\right]^{1/2} = \left[X^2 + 4k'^2X^2(1 - X)^2 + O\left(k'^4\right)\right]^{1/2}\\ &= X + 2k'^2X(1 - X)^2 + O\left(k'^4\right)\end{aligned}\right\} \tag{B8}$$

where equation (B2) has been used for Y^2. Thus, by retaining only terms of the order of $k'^2 \ln k'$, equations (B6) and (B7) become

APPENDIX B

$$\text{Logarithmic component of } J_2 = 2\sqrt{X_0(1 - X_0)}\,(1 - 2X_0)\,k'^2 \ln k' + O(k'^3 \ln k')$$

$$= 2\sqrt{X_0(1 - X_0)}\,(1 - 2X_0)\,k'^2 \ln k' \left[(y - \eta)^2 \frac{dk'}{d\eta}\right]\left[\frac{d\eta}{dk'} \frac{1}{(y - \eta)^2}\right] + O(k'^3 \ln k')$$

and

$$\text{Logarithmic component of } J_1 = 2\sqrt{X_0(1 - X_0)}\,(1 - X_0)\,k'^2 \ln k' + O(k'^3 \ln k')$$

$$= 2\sqrt{X_0(1 - X_0)}\,(1 - X_0)\,k'^2 \ln k' \left[\frac{dk'}{d\eta}(y - \eta)^2\right]\left[\frac{d\eta}{dk'} \frac{1}{(y - \eta)^2}\right] + O(k'^3 \ln k')$$

where X_0, the value of X at $\eta = y$, has been used to replace X. The derivative $dk'/d\eta$ is evaluated in appendix A. It is known from equations (A14) and (A18) that

$$\frac{dk'}{d\eta} \approx \frac{-1}{4k'A^2B^2} \frac{\beta^2(y - \eta)}{4\,c^2(\eta)} + O(k')$$

From equation (B2),

$$Y^2 = \frac{\beta^2(y - \eta)^2}{4\,c^2(\eta)} = 4k'^2X^2(1 - X)^2\left[1 + O(k'^2)\right]$$

or

$$y - \eta = \frac{\text{sign } (y - \eta)\ 4\ c(\eta)\ k'X(1 - X)}{\beta}\left[1 + O(k'^2)\right]$$

$$\approx \frac{\text{sign } (y - \eta)}{\beta}\ 4\ c(y)\ k'X_0(1 - X_0)\left[1 + O(k'^2)\right]$$

Hence,

APPENDIX B

$$\frac{d\eta}{dk'} = -\frac{4k'X_0^2(1-X_0)^2}{1} \frac{4c^2(y)\beta}{\beta^2 4c(y)k'X_0(1-X_0)\ \text{sign}(y-\eta)\left[1+O(k'^2)\right]} + O(k')$$

$$= -\frac{\text{sign}(y-\eta)}{\beta} 4c(y)X_0(1-X_0) + O(k') \tag{B9}$$

Furthermore,

$$\frac{d\eta}{dk'}\frac{1}{(y-\eta)^2} = -\frac{\text{sign}(y-\eta)}{\beta} 4c(y)X_0(1-X_0) \frac{\beta^2}{16c^2(y)k'^2X_0^2(1-X_0)^2} + O\left(\frac{1}{k'}\right)$$

$$= -\text{sign}(y-\eta)\frac{\beta}{4c(y)k'^2X_0(1-X_0)} + O\left(\frac{1}{k'}\right)$$

It follows that

$$\text{Logarithmic component of } J_2 = 2\sqrt{X_0(1-X_0)}(1-2X_0)k'^2 \ln k' (y-\eta)^2 \frac{dk'}{d\eta}\left[-\text{sign}(y-\eta)\frac{\beta}{4c(y)k'^2X_0(1-X_0)} + O\left(\frac{1}{k'}\right)\right] + O(k'^3 \ln k')$$

$$= -\text{sign}(y-\eta)\beta\frac{1-2X_0}{2c(y)\sqrt{X_0(1-X_0)}}(y-\eta)^2\frac{dk'}{d\eta}\ln k' + O(k'^3 \ln k')$$

$$= -\text{sign}(y-\eta)\frac{\beta}{c(y)}\frac{\cos\phi}{\sin\phi}(y-\eta)^2\frac{dk'}{d\eta}\ln k' + O(k'^3 \ln k') \tag{B10}$$

Similarly,

APPENDIX B

$$\text{Logarithmic component of } J_1 = 2(1 - X_0)\sqrt{X_0(1 - X_0)}\, k'^2 \ln k' \frac{dk'}{d\eta} (y - \eta)^2 \left[-\text{sign}\,(y - \eta) \frac{\beta}{4\, c(y)\, k'^2 X_0 (1 - X_0)} \right] + O(k'^3 \ln k')$$

$$= -\text{sign}\,(y - \eta) \frac{\beta}{2\, c(y)} \sqrt{\frac{1 - X_0}{X_0}}\, (y - \eta)^2 \frac{dk'}{d\eta} \ln k' + O(k'^3 \ln k')$$

$$= -\text{sign}\,(y - \eta) \frac{\beta}{2\, c(y)} \frac{1 + \cos\phi}{\sin\phi} (y - \eta)^2 \frac{dk'}{d\eta} \ln k' + O(k'^3 \ln k') \qquad \text{(B11)}$$

If equations (B10) and (B11) are substituted into equation (14), the logarithmic component of I_j can be shown to be

$$L_j = \frac{1}{\pi}\left[g_1(\phi) + (1 - X)\, g_2(\phi)\right]\left[-\text{sign}\,(y - \eta) \frac{\beta}{c(y)} \frac{\cos\phi}{\sin\phi} (y - \eta)^2 \frac{dk'}{d\eta} \ln k' + O(k'^3 \ln k')\right]$$

$$- \frac{g_2(\phi)}{\pi}\left[-\text{sign}\,(y - \eta) \frac{\beta}{2\, c(y)} \frac{1 + \cos\phi}{\sin\phi} (y - \eta)^2 \frac{dk'}{d\eta} \ln k' + O(k'^3 \ln k')\right]$$

$$= \text{sign}\,(y - \eta)\, \beta \frac{(y - \eta)^2 \frac{dk'}{d\eta} \ln k'}{\pi\, c(y)} \left(\left\{ \cos j\phi + \cos(j + 1)\phi - \frac{1}{2} \frac{(1 + \cos\phi)}{\sin\phi} 2\left[j \sin j\phi \right.\right.\right.$$

$$\left.\left.\left. + (j + 1) \sin(j + 1)\phi \right] \right\} \left(-\frac{\cos\phi}{\sin\phi} \right) - \frac{j \sin j\phi + (j + 1) \sin(j + 1)\phi}{\sin\phi} \frac{1 + \cos\phi}{\sin\phi} \right)$$

$$= -\text{sign}\,(y - \eta) \frac{\beta}{\pi} \frac{(y - \eta)^2 \frac{dk'}{d\eta} \ln k'}{c(y)} G_j(\phi) \qquad \text{(B12)}$$

where

$$G_j(\phi) = \frac{\cos j\phi + \cos(j + 1)\phi}{\sin\phi} \cos\phi + j \sin j\phi + (j + 1) \sin(j + 1)\phi \qquad \text{(B13)}$$

It should be noted that equation (B12) can be reduced to the conventional form, such as that used by Wagner (ref. 5), by simply using equation (B9) for $dk'/d\eta$:

APPENDIX B

$$\frac{dk'}{d\eta} = -\text{sign}\,(y - \eta)\,\frac{\beta}{4\,c(y)\,X_0(1 - X_0)}$$

$$= -\text{sign}\,(y - \eta)\,\frac{\beta}{c(y)\,\sin^2\phi}$$

and to the first approximation,

$$\ln k' = \ln\sqrt{\frac{Y^2}{4X^2(1 - X)^2}}$$

$$= \ln|Y| - \ln\left[2X_0(1 - X_0)\right]$$

If only the term associated with $\ln|Y|$ is retained, it is seen from equation (B12) that

$$L_j = -\text{sign}\,(y - \eta)\,\frac{\beta}{\pi}\,\frac{(y - \eta)^2}{c(y)}\ln|Y|\left[-\text{sign}\,(y - \eta)\,\frac{\beta}{c(y)\,\sin^2\phi}\right]G_j(\phi)$$

$$= \frac{4}{\pi}|Y|^2\ln|Y|\,\frac{G_j(\phi)}{\sin^2\phi} + O\left(|Y|^3\ln|Y|\right)$$

where $c(y)$ has been expressed in terms of $c(\eta)$ through the relation

$$c(\eta) = c(y) + c'(y)\,(\eta - y) + O(\eta - y)^2$$

Since

$$\left.\frac{dh_j(\phi_1)}{dX_i}\right|_{\phi_1=\phi} = -2\left[\frac{\cos j\phi + \cos(j + 1)\phi}{\sin^3\phi}\cos\phi + \frac{j\sin j\phi + (j + 1)\sin(j + 1)\phi}{\sin^2\phi}\right]$$

$$= -2\,\frac{G_j(\phi)}{\sin^2\phi}$$

it follows that

$$L_j = -\frac{2}{\pi}\left.\frac{dh_j(\phi_1)}{dX_1}\right|_{\phi_1=\phi}|Y|^2\ln|Y|$$

which is the same expression as that used by Wagner (eq. (16b) of ref. 5) except for the factor $2/\pi$ which is due to the different definition of $h_j(\phi_1)$ used here in this report.

To perform the spanwise integration defined in equation (5), the chordwise influence function $I_j(X,Y,\eta)$ and the loading function $q_j(\eta)\, I_j(X,Y,\eta)$ may be decomposed as follows:

$$I_j(X,Y,\eta) = I_j^*(X,Y,\eta) + L_j \tag{B14}$$

$$q_j(\eta)\, I_j(X,Y,\eta) = q_j(y)\left(I_j - I_j^*\right) + \left[q_j(\eta) - q_j(y)\right] I_j + q_j(y)\, I_j^* \tag{B15}$$

When equations (B14) and (B15) are substituted into the spanwise integral in equation (5), the first term in equation (B15) can be exactly integrated and gives

$$w\left(\phi_s, y_\nu\right)_{\text{part 1}} = -\frac{1}{4\pi} \sum_{j=0}^{N-1} \int_{-1}^{1} \frac{q_j(y_\nu)}{(y_\nu - \eta)^2} L_j \, d\eta$$

$$= \frac{\beta}{4\pi^2} \sum_{j=0}^{N-1} q_j(y_\nu) \frac{G_j(\phi_s)}{c(y_\nu)} \left(\int_{-1}^{y_\nu} \frac{dk'}{d\eta} \ln k' \, d\eta - \int_{y_\nu}^{1} \frac{dk'}{d\eta} \ln k' \, d\eta \right)$$

$$= \frac{\beta}{4\pi^2} \sum_{j=0}^{N-1} q_j(y_\nu) \frac{G_j(\phi_s)}{c(y_\nu)} \left(\int_{k_1'}^{0} \ln k' \, dk' - \int_{0}^{k_2'} \ln k' \, dk' \right)$$

$$= -\frac{\beta}{4\pi^2} \sum_{j=0}^{N-1} q_j(y_\nu) \frac{G_j(\phi_s)}{c(y_\nu)} \left(k_1' \ln k_1' - k_1' + k_2' \ln k_2' - k_2' \right)$$

The notation has been made compatible with Multhopp's notation (ref. 1). Reference 7 gives a detailed explanation of the subscripts which have been introduced. It follows that

$$-2\pi\, w\left(\phi_s, y_\nu\right)_{\text{part 1}} = -\sum_{j=0}^{N-1} \frac{\beta\, q_j(y_\nu)}{2\pi\, c(y_\nu)} G_j(\phi_s) \left(k_1' \ln k_1' - k_1' + k_2' \ln k_2' - k_2' \right) \tag{B16}$$

where

$$k_1'^2 = \frac{(A_1 + B_1)^2 - 1}{4A_1B_1}$$

$$A_1^2 = (1 - X_1)^2 + Y_1^2$$

$$B_1^2 = X_1^2 + Y_1^2$$

$$X_1 = \frac{1}{2}\frac{\xi_{le}(y_\nu) + c(y_\nu)(1 - \cos\phi_s) - \xi_{le}(-1)}{c(-1)}$$

$$Y_1 = \frac{\beta}{2}\frac{y + 1}{c(-1)}$$

$$k_2'^2 = \frac{(A_2 + B_2)^2 - 1}{4A_2B_2}$$

$$A_2^2 = (1 - X_2)^2 + Y_2^2$$

$$X_2 = \frac{1}{2}\frac{\xi_{le}(y_\nu) + c(y_\nu)(1 - \cos\phi_s) - \xi_{le}(1)}{c(1)}$$

$$Y_2 = \frac{\beta}{2}\frac{y - 1}{c(1)}$$

(B17)

If $c(-1) = c(1) = 0$ (i.e., the taper ratio is zero), the limiting values for k_1' and k_2' must be used:

APPENDIX B

$$\lim_{c(-1)\to 0} k_1'^2 = \lim_{c(-1)\to 0} \frac{\frac{1}{4\,c^2(-1)}\left\{2\sqrt{\left[d(y_\nu) - c(y_\nu)\cos\phi_s - d(-1)\right]^2 + \beta^2(y_\nu+1)^2}\right\}^2}{4\,\frac{1}{4\,c^2(-1)}\left\{\left[d(y_\nu) - c(y_\nu)\cos\phi_s - d(-1)\right]^2 + \beta^2(y_\nu+1)^2\right\}} = 1$$

where $d(y_\nu)$ is the x-location of the midchord at y_ν. Similarly, $\lim_{c(+1)\to 0} k_2'^2 = 1$. It follows that

$$k_1' \ln k_1' - k_1' + k_2' \ln k_2' - k_2' = -2 \tag{B18}$$

for a symmetrical wing in which $c(1) = 0$ and $c(-1) = 0$.

The spanwise integration of the second and third terms in equation (B15) can be accomplished by Multhopp's method of integration. Substitution of equation (B12) for L_j results in

$$-2\pi\, w(\phi_s, y_\nu)_{\text{part 2}} = -\frac{1}{2\pi}\sum_{j=0}^{N-1} \int_{-1}^{1} \pi\left[q_j(\eta)\, I_j - q_j(y_\nu)\, L_j\right]\frac{d\eta}{(y_\nu - \eta)^2}$$

$$= -\frac{1}{2\pi}\sum_{j=0}^{N-1} \int_{-1}^{1} \pi\left[q_j(\eta)\, I_j\right.$$

$$\left. + \operatorname{sign}(y_\nu - \eta)\,\frac{\beta}{\pi}\, q_j(y_\nu)\,\frac{G_j(\phi_s)}{c(y_\nu)}\,(y_\nu - \eta)^2\,\frac{dk'}{d\eta}\ln k'\right]\frac{d\eta}{(y_\nu - \eta)^2}$$

Multhopp's method of integration can be applied by using the trigonometric interpolation formula of the type

$$q_j(\eta)\, I_j(\eta) = \frac{2}{m+1}\sum_{n=-\frac{m-1}{2}}^{\frac{m-1}{2}} (q_j I_j)_n \sum_{\lambda=1}^{m} \sin\lambda\theta_n \sin\lambda\theta$$

where $\theta_n = \frac{\pi}{2} - \frac{n\pi}{m+1}$ and m is the number of spanwise integration stations over the whole wing. Thus,

$$-2\pi\, w\left(\phi_s, y_\nu\right)_{\text{part 2}} = \sum_{j=0}^{N-1} \left\{ b_{\nu\nu}\, q_j\left(y_\nu\right)\, \pi I_{j\nu\nu} - \sum_{n}{}' \, b_{\nu n} \left[q_j\left(\eta_n\right)\, \pi I_{j\nu n} + \text{sign}\left(y_\nu - \eta_n\right)\, \beta\, q_j\left(y_\nu\right)\, \frac{G_j\left(\phi_s\right)}{c\left(y_\nu\right)} \left(y_\nu - \eta_n\right)^2 \frac{dk'_{\nu n}}{d\eta} \ln k'_{\nu n} \right] \right\} \tag{B19}$$

where the prime on the summation sign means that the term with $n = \nu$ is omitted from the summation and

$$\left.\begin{aligned}
&\frac{dk'_{\nu n}}{d\eta} = \frac{k'_{\nu n}}{2A^2B^2}\left[-X(A-B)^2 + B^2 - AB\right]\frac{dX}{d\eta} + \frac{\beta\left(y_\nu - \eta_n\right)}{8k'_{\nu n}A^2B^2\, c\left(\eta_n\right)}\left[1 - 2k'^2_{\nu n}(A-B)^2\right]\frac{dY}{d\eta} \\
&X = \frac{1}{2}\,\frac{\ell_{le}\left(y_\nu\right) + c\left(y_\nu\right)\left(1 - \cos\phi_s\right) - \ell_{le}\left(\eta_n\right)}{c\left(\eta_n\right)} \\
&Y = \frac{1}{2}\,\frac{\beta\left(y_\nu - \eta_n\right)}{c\left(\eta_n\right)} \\
&A^2 = (1-X)^2 + Y^2 \\
&B^2 = X^2 + Y^2 \\
&\frac{dX}{d\eta} = -\frac{1}{2}\left\{\frac{\tan\Lambda_{le}\left(\eta_n\right)}{c\left(\eta_n\right)} + \frac{\left[\ell_{le}\left(y_\nu\right) + c\left(y_\nu\right)\left(1 - \cos\phi_s\right) - \ell_{le}\left(\eta_n\right)\right]\left[\tan\Lambda_{te}\left(\eta_n\right) - \tan\Lambda_{le}\left(\eta_n\right)\right]}{2\, c^2\left(\eta_n\right)}\right\} \\
&\frac{dY}{d\eta} = -\frac{\beta}{2}\left\{\frac{1}{c\left(\eta_n\right)} + \frac{\left(y_\nu - \eta_n\right)\left[\tan\Lambda_{te}\left(\eta_n\right) - \tan\Lambda_{le}\left(\eta_n\right)\right]}{2\, c^2\left(\eta_n\right)}\right\} \\
&k'^2_{\nu n} = \frac{(A+B)^2 - 1}{4AB} \\
&G_j\left(\phi_s\right) = \frac{\cos j\phi_s + \cos(j+1)\,\phi_s}{\sin\phi_s}\cos\phi_s + j\sin j\phi_s + (j+1)\sin(j+1)\,\phi_s
\end{aligned}\right\} \tag{B20}$$

Also

$$\left.\begin{aligned} b_{\nu\nu} &= \frac{m+1}{4 \sin \theta_\nu} \\ b_{\nu n} &= \frac{\sin \theta_n}{(m+1)\left(\cos \theta_n - \cos \theta_\nu\right)^2} \qquad \left(|n-\nu| \text{ odd}\right) \\ &= 0 \qquad \left(|n-\nu| \text{ even}\right) \end{aligned}\right\} \tag{B21}$$

$$\left.\begin{aligned} I_{0\nu\nu} &= \frac{2}{\pi}\left(\phi_s + \sin \phi_s\right) \\ I_{j\nu\nu} &= \frac{2}{\pi}\left[\frac{\sin j\phi_s}{j} + \frac{\sin (j+1)\,\phi_s}{j+1}\right] \qquad (j \geq 1) \end{aligned}\right\} \tag{B22}$$

Combination of equations (B16) and (B19) gives the total induced downwash:

$$\begin{aligned} -2\pi w\left(\phi_s, y_\nu\right) = &\sum_{j=0}^{N-1} \left\{ q_j\left(y_\nu\right)\left[b_{\nu\nu}\pi I_{j\nu\nu} - \beta \frac{G_j\left(\phi_s\right)}{2\pi\, c\left(y_\nu\right)}\left(k'_1 \ln k'_1 - k'_1 + k'_2 \ln k'_2 - k'_2\right) \right.\right. \\ &\left.\left. - \sum_n{}' b_{\nu n} \beta \operatorname{sign}\left(y_\nu - \eta_n\right) \frac{G_j\left(\phi_s\right)}{c\left(y_\nu\right)}\left(y_\nu - \eta_n\right)^2 \frac{dk'_{\nu n}}{d\eta} \ln k'_{\nu n} \right] - \sum_n{}' b_{\nu n}\, q_j\left(\eta_n\right)\, \pi I_{j\nu n} \right\} \\ = &\sum_{j=0}^{N-1} \left\{ q_j\left(y_\nu\right)\left[b_{\nu\nu}\pi I_{j\nu\nu} - \beta \frac{G_j\left(\phi_s\right)}{2\pi\, c\left(y_\nu\right)}\left(k'_1 \ln k'_1 - k'_1 + k'_2 \ln k'_2 - k'_2\right) \right.\right. \\ &\left.\left. - \sum_n{}' \beta \frac{\sin \theta_n}{m+1} \frac{G_j\left(\phi_s\right)}{c\left(y_\nu\right)} \frac{dk'_{\nu n}}{d\eta} \ln k'_{\nu n} \right] - \sum_n{}' b_{\nu n}\, q_j\left(\eta_n\right)\, \pi I_{j\nu n} \right\} \end{aligned} \tag{B23}$$

APPENDIX C

IMPROVEMENTS IN LANGLEY PROGRAM A0313

The improvement in the logarithmic-singularity correction term presented in this report has been implemented into the analysis version of the computer program described in the supplement to reference 7 (Langley program A0313). In addition,

(1) The program has been restructured into an overlay arrangement (core requirements of 51000_8 words on the Control Data 6600 computer system).

(2) The chordal loading functions of reference 5 are employed.

(3) The test to set the values for N and m is utilized before a solution is begun.

(4) Damping in roll, damping in pitch, and side-edge suction-force computations have been added. In order to access these items, the format of the third data input card has been changed to 5F6.0, F6.2, and 4F6.0 with the last three fields set aside to receive the codes needed to commence these computations. The codes are as follows:

(a) In columns 43 to 48, a 1 causes the damping-in-roll stability derivative C_{l_p} to be computed; a 0 indicates that C_{l_p} is not desired.

(b) In columns 49 to 54, a 1 causes the damping-in-pitch stability derivative C_{m_q} and the lift coefficient due to pitch rate C_{L_q} to be computed; a 0 indicates that C_{m_q} and C_{L_q} are not required.

(c) In columns 55 to 60, a 1 causes the side-edge suction force to be computed and the aerodynamic characteristics as a function of angle of attack to be listed; a 0 indicates that neither the side-edge suction force nor a listing of the aerodynamic characteristics is required.

Note that roll-rate and pitch-rate stability derivatives cannot be computed simultaneously. Also, the side-edge suction force should not be computed concurrently with roll-rate or pitch-rate derivatives.

This version of the computer program is available from COSMIC, 112 Barrow Hall, University of Georgia, Athens, GA 30602.

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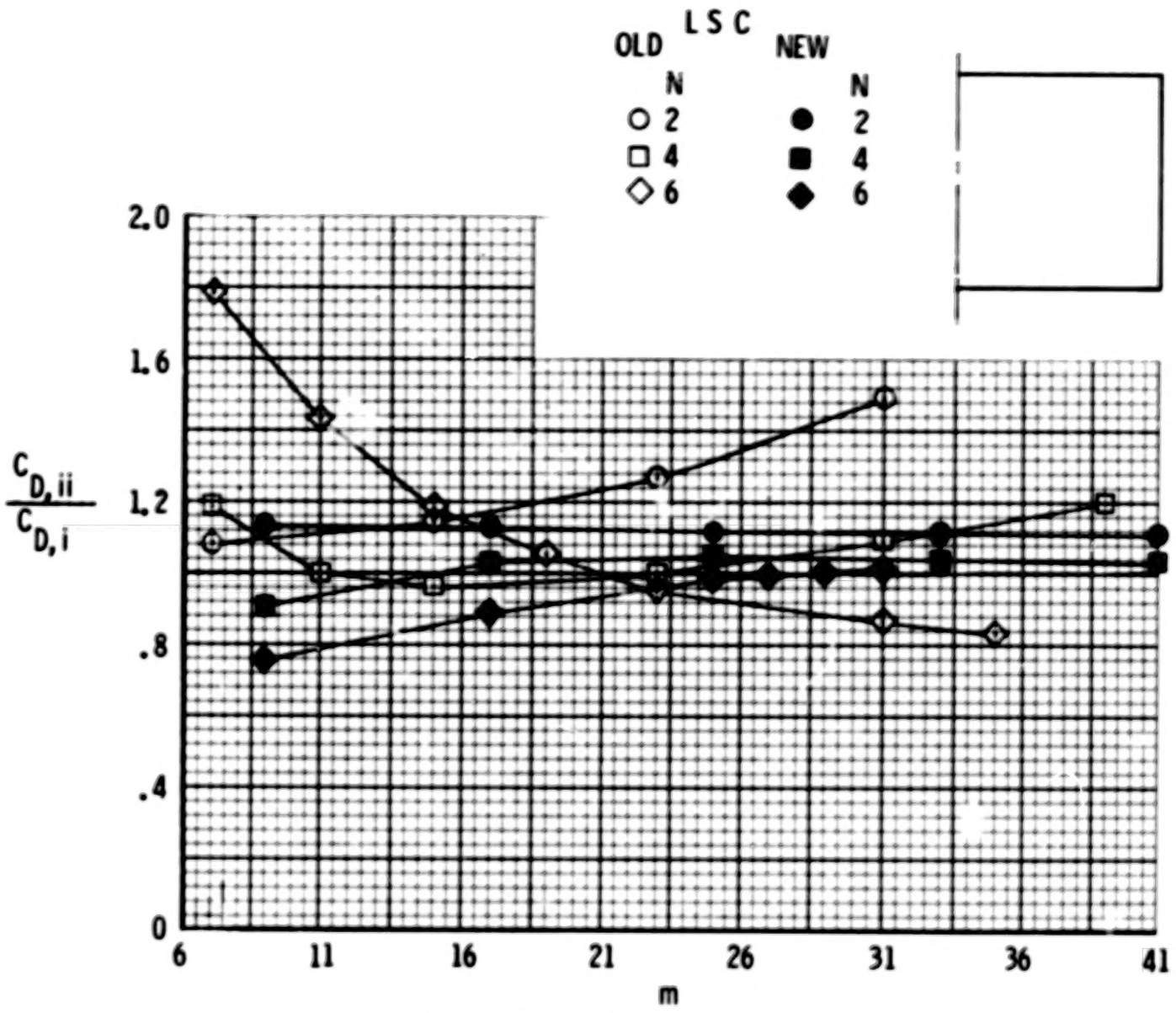
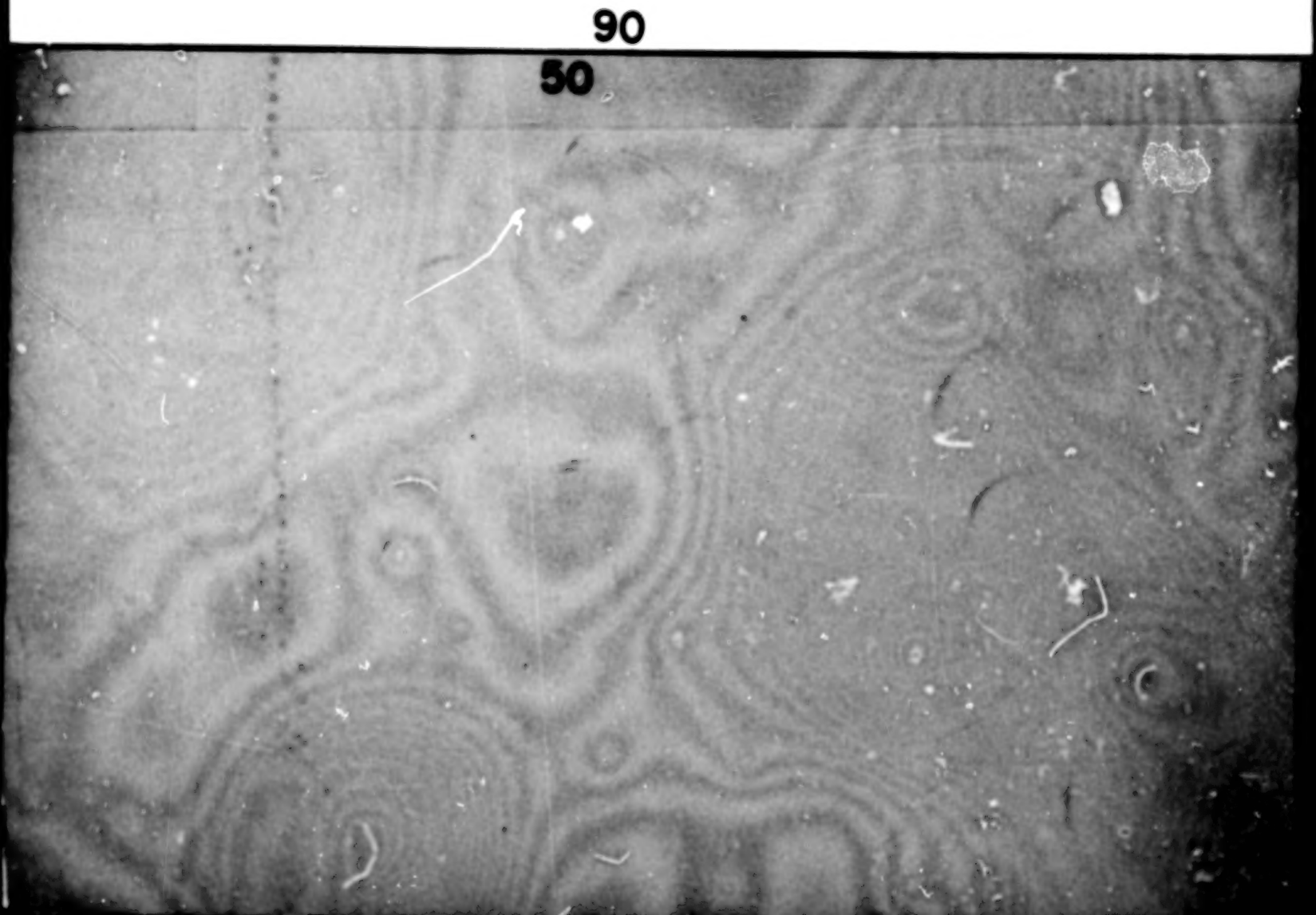


(a) Aspect-ratio-2 rectangular wing.

Figure 1.- Effect of LSC on near-field to far-field vortex drag ratio for two planforms at various values of N. M = 0.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